

# A survey on the inverse integrating factor.\*

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## Abstract

The relation between limit cycles of planar differential systems and the inverse integrating factor was first shown in an article of Giacomini, Llibre and Viano appeared in 1996. From that moment on, many research articles are devoted to the study of the properties of the inverse integrating factor and its relation with limit cycles and their bifurcations. This paper is a summary of all the results about this topic. We include a list of references together with the corresponding related results aiming at being as much exhaustive as possible. The paper is, nonetheless, self-contained in such a way that all the main results on the inverse integrating factor are stated and a complete overview of the subject is given. Each section contains a different issue to which the inverse integrating factor plays a role: the integrability problem, relation with Lie symmetries, the center problem, vanishing set of an inverse integrating factor, bifurcation of limit cycles from either a period annulus or from a monodromic  $\omega$ -limit set and some generalizations.

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## 1 The Euler integrating factor

The method of integrating factors is, in principle, a means for solving ordinary differential equations of first order and it is theoretically important. The use of integrating factors goes back to Leonhard Euler.

Let us consider a first order differential equation and write the equation in the Pfaffian form

$$\omega = P(x, y) dy - Q(x, y) dx = 0 . \quad (1)$$

We assume that the functions  $P$  and  $Q$  are of class  $\mathcal{C}^1$  in a region  $\mathcal{U} \subseteq \mathbb{R}^2$ . If there is a solution of (1) which may be expressed in the form  $H(x, y) = h$  with  $H$  having

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continuous partial derivatives in  $\mathcal{U}$  and with  $h$  an arbitrary constant, then it is not difficult to see that such an  $H$  satisfies the linear partial differential equation

$$P \frac{\partial H}{\partial x} + Q \frac{\partial H}{\partial y} = 0 . \quad (2)$$

Conversely, every non-constant solution  $H$  of (2) gives also a solution  $H(x, y) = h$  of (1). Thus, solving (1) and solving (2) are equivalent tasks.

It is straightforward to show that if  $H_0(x, y)$  is a non-constant solution of equation (2), then all solutions of this equation are of the form  $F(H_0(x, y))$  where  $F$  is a freely chosen function with continuous derivative. The connection between equations (1) and (2) may be presented also in another form. Suppose that  $H(x, y) = h$  is any solution of (1). Then (2) implies

$$\frac{\partial H / \partial y}{P} = - \frac{\partial H / \partial x}{Q} .$$

If we denote the common value of these two ratios by  $\mu(x, y)$ , then we have  $\partial H / \partial y = \mu P$  and  $\partial H / \partial x = -\mu Q$ . This gives to the differential of the function  $H$  the expression  $dH(x, y) = \mu(x, y)(P(x, y) dy - Q(x, y) dx)$ . Hence,  $\mu(x, y)$  is called the integrating factor of the given differential equation (1) because the left hand side of (1) turns, when multiplied by  $\mu(x, y)$ , to be an exact differential.

Conversely, any integrating factor  $\mu$  of (1), i.e. such that  $\mu(x, y)(P(x, y) dy - Q(x, y) dx)$  is the differential of some function  $H$ , is easily seen to determine the solutions of the form  $H(x, y) = h$  of (1). Altogether, solving the differential equation (1) is equivalent to finding an integrating factor of the equation.

When an integrating factor  $\mu$  of (1) is available, the function  $H$  can be obtained from the line integral

$$H(x, y) = \int_{(x_0, y_0)}^{(x, y)} \mu(x, y)(P(x, y) dy - Q(x, y) dx)$$

along any curve connecting an arbitrarily chosen point  $(x_0, y_0)$  and the point  $(x, y)$  in the region  $\mathcal{U}$ . We remark that this line integral might not be well-defined if the region  $\mathcal{U}$  is not simply-connected. When we know an integrating factor  $\mu$  of (1), we have a first integral well-defined in each simply-connected subcomponent of the region  $\mathcal{U}$ .

## 2 The inverse integrating factor

Let us consider a real planar autonomous differential system

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \quad (3)$$

where  $P(x, y)$  and  $Q(x, y)$  are of class  $\mathcal{C}^1(\mathcal{U})$  and  $\mathcal{U} \subseteq \mathbb{R}^2$  is an open set. The dot denotes derivation with respect to the independent variable  $t$  usually called *time*, that is  $\dot{\phantom{x}} = \frac{d}{dt}$ .

As usual, we associate to system (3) the vector field  $\mathcal{X} = P(x, y)\partial_x + Q(x, y)\partial_y$ . Notice that the ordinary differential equation  $\omega = 0$  given in (1) is just the differential equation of the orbits of system (3).

**Definition 1** *A function  $V : \mathcal{U} \rightarrow \mathbb{R}$  is said to be an inverse integrating factor of system (3) if it is of class  $\mathcal{C}^1(\mathcal{U})$ , it is not locally null and it satisfies the following partial differential equation:*

$$P(x, y) \frac{\partial V(x, y)}{\partial x} + Q(x, y) \frac{\partial V(x, y)}{\partial y} = \left( \frac{\partial P(x, y)}{\partial x} + \frac{\partial Q(x, y)}{\partial y} \right) V(x, y). \quad (4)$$

In short notation, an inverse integrating factor  $V$  of system (3) satisfies  $\mathcal{X}V = V \operatorname{div} \mathcal{X}$ , where  $\operatorname{div} \mathcal{X} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$  stands for the divergence of the vector field  $\mathcal{X}$ .

Of course, the computation of an inverse integrating factor for a concrete system is a delicate matter whose difficulty is comparable to solving the system itself.

If  $V$  is an inverse integrating factor of a  $\mathcal{C}^1$  vector field  $\mathcal{X}$ , then the zero set of  $V$ ,  $V^{-1}(0) := \{(x, y) \mid V(x, y) = 0\}$ , is composed of trajectories of  $\mathcal{X}$ . For by the equation (4) that defines  $V$ ,  $\mathcal{X}$  is orthogonal to the gradient vector field  $\nabla V$  along the zero set of  $V$ .

The name “inverse integrating factor” arises from the fact that if  $V$  solves equation (4), then its reciprocal  $1/V$  is an integrating factor for  $\mathcal{X}$  on  $\mathcal{U} \setminus V^{-1}(0)$ .

### 3 Local nontrivial Lie symmetries and inverse integrating factors

Roughly speaking, a symmetry group of a system of differential equations is a continuous group which transforms solutions of the system to other solutions. Simple typical examples are groups of translations, rotations and scalings, but these certainly do not exhaust the range of possibilities. Once one has determined the symmetry group of a system of differential equations, a number of applications become available.

More precisely, a symmetry of system (3) in  $\mathcal{U}$ , where  $\mathcal{U} \subseteq \mathbb{R}^2$  is an open set, is a 1-parameter Lie group of diffeomorphisms  $\Phi_\epsilon$  acting in  $\mathcal{U}$  that maps the set of orbits of (3) into itself. When  $\Phi_\epsilon(x, y) = (\bar{x}(x, y; \epsilon), \bar{y}(x, y; \epsilon))$ , the symmetry condition of (3) reads for  $\bar{x} = P(\bar{x}, \bar{y})$ ,  $\bar{y} = Q(\bar{x}, \bar{y})$  for all  $\epsilon$  close to zero. Let the  $\mathcal{C}^1(\mathcal{U})$  vector field  $\mathcal{Y} = \xi(x, y)\partial_x + \eta(x, y)\partial_y$  be the infinitesimal generator of the 1-parameter Lie group  $\Phi_\epsilon$ , that is,  $\bar{x}(x, y; \epsilon) = x + \epsilon\xi(x, y) + O(\epsilon^2)$ ,  $\bar{y}(x, y; \epsilon) = y + \epsilon\eta(x, y) + O(\epsilon^2)$ .

Denoting by  $\mathcal{X} = P(x, y)\partial_x + Q(x, y)\partial_y$  the vector field associated to system (3), it is well known that a characterization of the Lie symmetries of (3) is given by the relation  $[\mathcal{X}, \mathcal{Y}] = \mu(x, y)\mathcal{X}$  for certain scalar function  $\mu : \mathcal{U} \rightarrow \mathbb{R}$ . In this expression we have used the *Lie bracket* of two  $\mathcal{C}^1$ -vector fields  $\mathcal{X}$  and  $\mathcal{Y}$  defined as  $[\mathcal{X}, \mathcal{Y}] := \mathcal{X}\mathcal{Y} - \mathcal{Y}\mathcal{X}$ . Using coordinates we have

$$[\mathcal{X}, \mathcal{Y}] = \left( P \frac{\partial \xi}{\partial x} - \xi \frac{\partial P}{\partial x} + Q \frac{\partial \xi}{\partial y} - \eta \frac{\partial P}{\partial y} \right) \partial_x + \left( P \frac{\partial \eta}{\partial x} - \xi \frac{\partial Q}{\partial x} + Q \frac{\partial \eta}{\partial y} - \eta \frac{\partial Q}{\partial y} \right) \partial_y . \quad (5)$$

When beginning students first encounter ordinary differential equations, they are presented with a variety of special techniques designed to solve certain particular types of equations, such as separable, homogeneous or exact. Indeed, this was the state of the art around the middle of the nineteenth century, when Sofus Lie made the profound discovery that these special methods were, in fact, all special cases of a general integration procedure based on the invariance of the differential equation under a continuous group of symmetries. This observation at once unified and significantly extended the available integration techniques.

DIFFERENTIAL EQUATION	LIE SYMMETRY
$dy/dx = f(x)g(y)$	$\mathcal{Y} = g(y)\partial_y$
$dy/dx = f(ax + by)$	$\mathcal{Y} = b\partial_x + a\partial_y$
$dy/dx = \frac{y+xf(\sqrt{x^2+y^2})}{x-yf(\sqrt{x^2+y^2})}$	$\mathcal{Y} = y\partial_x - x\partial_y$
$dy/dx = f(y/x)$	$\mathcal{Y} = x\partial_x + y\partial_y$
$dy/dx = P(x)y + Q(x)$	$\mathcal{Y} = \exp\left(\int P(x)dx\right)\partial_y$
$dy/dx = P(x)y + Q(x)y^n$	$\mathcal{Y} = y^n \exp\left[(1-n)\int P(x)dx\right]\partial_y$

Consider now a  $\mathcal{C}^1$  vector field  $\mathcal{X} = P(x, y)\partial_x + Q(x, y)\partial_y$  defined in an open connected subset  $\mathcal{U} \subseteq \mathbb{R}^2$ . In the case of a single first order ordinary differential equation  $dy/dx = Q(x, y)/P(x, y)$ , the Lie symmetries method provides by quadrature an explicit formula for the general solution. In fact, one can easily see that if we know a Lie symmetry in  $\mathcal{U}$  with infinitesimal generator  $\mathcal{Y} = \xi(x, y)\partial_x + \eta(x, y)\partial_y$  then we construct an inverse integrating factor  $V = \det\{\mathcal{X}, \mathcal{Y}\} = P\eta - Q\xi$  defined in  $\mathcal{U}$ , but the converse is not always true. To see that, assume now the existence of an inverse integrating factor  $V$  of  $\mathcal{X}$  in a simply connected domain  $\mathcal{U}$  and we look for an infinitesimal generator  $\mathcal{Y} = \xi(x, y)\partial_x + \eta(x, y)\partial_y$  of a Lie symmetry of  $\mathcal{X}$  well defined in  $\mathcal{U}$ . We recall that a singular point  $p \in U$  of  $\mathcal{X}$  is called *weak* if  $\text{div}\mathcal{X}(p) = 0$ . If there is no weak singularity of  $\mathcal{X}$  in  $\mathcal{U}$ , then we can do at least one of the following constructions:

- (i) Prescribe the function  $\xi(x, y)$  and solve  $\eta(x, y)$  from  $V = P\eta - Q\xi$ .

(ii) Prescribe the function  $\eta(x, y)$  and solve  $\xi(x, y)$  from  $V = P\eta - Q\xi$ .

(iii) Take the rescaled hamiltonian vector field

$$\mathcal{Y} = \frac{1}{\operatorname{div}\mathcal{X}} \left( -\frac{\partial V}{\partial y} \partial_x + \frac{\partial V}{\partial x} \partial_y \right),$$

defined in  $\mathcal{U} \setminus \{(x, y) \in \mathcal{U} : \operatorname{div}\mathcal{X} = 0\}$ .

Therefore, the equivalence between inverse integrating factors and Lie symmetries for planar vector fields  $\mathcal{X}$  is not true, in general, in neighborhoods of weak singular points of  $\mathcal{X}$ . Of course, some special situations can appear giving the equivalence when  $\mathcal{X}$  possesses an analytic first integral in these neighborhoods as the nondegenerate center singular point shows.

Importance of inverse integrating factors arises from the fact that the differential 1-form  $\omega/V = (P dy - Q dx)/V$  is closed ( $d(\omega/V) = 0$ ) in  $\mathcal{U} \setminus V^{-1}(0)$ . Then in the case in which  $\mathcal{U} \setminus V^{-1}(0)$  is simply-connected, the 1-form  $\omega/V$  is *exact* ( $\omega/V = dH$ ), and therefore a  $\mathcal{C}^2$  first integral  $H(x, y)$  of the differential equation is immediately constructed. As a consequence, the vector field  $\mathcal{X} = P(x, y)\partial_x + Q(x, y)\partial_y$  is topologically equivalent, in  $\mathcal{U}$ , to the hamiltonian vector field  $\mathcal{X}/V = \frac{\partial H}{\partial y} \partial_x - \frac{\partial H}{\partial x} \partial_y$ .

Making a pause in this exposition we now present an example. Let us consider the following cubic system

$$\dot{x} = P(x, y) = -y - x(x^2 + y^2 - 1), \quad \dot{y} = Q(x, y) = x - y(x^2 + y^2 - 1). \quad (6)$$

An inverse integrating factor for system (6) is given by  $V(x, y) = (x^2 + y^2)(x^2 + y^2 - 1)$ . Associated to him one has the first integral

$$H(x, y) = \frac{(x^2 + y^2 - 1)}{(x^2 + y^2)} \exp \left\{ 2 \arctan \left( \frac{y}{x} \right) \right\},$$

which is not continuous in  $(0, 0)$ . On the other hand, since the polar form of the system is  $\dot{r} = 2r^2(r^2 - 1)$ ,  $\dot{\varphi} = 1$  it is easy to check that the unit circle  $x^2 + y^2 - 1 = 0$  is the unique limit cycle of system (6). Let  $\mathcal{X} = P(x, y)\partial/\partial x + Q(x, y)\partial/\partial y$  be the vector field associated with system (6). From the symmetries point of view, since  $\mathcal{Y} = y\partial_x - x\partial_y$  satisfies  $[\mathcal{X}, \mathcal{Y}] \equiv 0$  we have that  $\mathcal{Y}$  is the infinitesimal generator of a Lie group admitted by system (6) which is just the  $SO(2)$  rotation group  $\bar{x} = x \cos \epsilon - y \sin \epsilon$ ,  $\bar{y} = x \sin \epsilon + y \cos \epsilon$ . Hence  $V(x, y) = \det\{\mathcal{X}, \mathcal{Y}\}$  is an inverse integrating factor of system (6). Notice that the only common integral curves for the vector fields  $\mathcal{X}$  and  $\mathcal{Y}$  are included in  $V^{-1}(0)$  and are just the separatrices of  $\mathcal{X}$ . This behavior will be explained in future sections.

## 4 On the integrability problem

The integrability problem is mainly related to planar **polynomial** differential systems of the form

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \quad (7)$$

where  $P(x, y), Q(x, y) \in \mathbb{R}[x, y]$  are coprime polynomials, that is, there is no non-constant polynomial which divides both  $P$  and  $Q$ . We call  $d$  the maximum degree of  $P$  and  $Q$  and we say that system (7) is of degree  $d$ . When  $d = 2$ , we say that (7) is a *quadratic system*.

If  $p$  is a point such that  $P(p) = Q(p) = 0$ , then we say that  $p$  is a *singular point* of system (7).

As we have already defined in Section 1, a  $\mathcal{C}^j$  function  $H : \mathcal{U} \rightarrow \mathbb{R}$  such that it is constant on each trajectory of (7) and it is not locally constant is called a *first integral* of system (7) of class  $j$  defined on  $\mathcal{U} \subseteq \mathbb{R}^2$ . The equation  $H(x, y) = h$  for a fixed  $h \in \mathbb{R}$  gives a set of trajectories of the system, but in an implicit way. When  $j \geq 1$ , these conditions are equivalent to  $P(x, y) \frac{\partial H}{\partial x} + Q(x, y) \frac{\partial H}{\partial y} = 0$  and  $H$  not locally constant. The problem of finding such a first integral and the functional class it must belong to is what we call the *integrability problem*.

To find an integrating factor or an inverse integrating factor for system (7) is closely related to finding a first integral for it. When considering the integrability problem we are also addressed to study whether an (inverse) integrating factor belongs to a certain given class of functions.

When a first integral  $H$  of system (7) is known, all the orbits of the system are contained in its domain of definition are given by the level sets  $H(x, y) = h$ . Thus, a natural strategy is to look for the determination of some of the orbits of the system and try to build a first integral with them. In particular, and since system (7) is polynomial, those orbits which are algebraic will be of special interest.

An *invariant curve* is a curve given by  $f(x, y) = 0$ , where  $f : \mathcal{U} \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$  is a  $\mathcal{C}^1$  function in the open set  $\mathcal{U}$ , non locally constant and such that there exists a  $\mathcal{C}^1$  function in  $\mathcal{U}$ , denoted by  $k(x, y)$  and called *cofactor*, which satisfies:

$$P(x, y) \frac{\partial f}{\partial x}(x, y) + Q(x, y) \frac{\partial f}{\partial y}(x, y) = k(x, y) f(x, y), \quad (8)$$

for all  $(x, y) \in \mathcal{U}$ . The notion of invariant curve was first introduced in [53]. The identity (8) can be rewritten by  $\mathcal{X}f = kf$ . We recall that  $\mathcal{X}f$  denotes the scalar product of the vector field  $\mathcal{X}$  and the gradient vector  $\nabla f$  related to  $f(x, y)$ , that is,  $\nabla f(x, y) = (\frac{\partial f}{\partial x}(x, y), \frac{\partial f}{\partial y}(x, y))$ . We will denote by  $\frac{df}{dt}$  or by  $\dot{f}$  the function  $\mathcal{X}f$  once evaluated on a solution of system (7). In case  $f(x, y) = 0$  defines a curve in the real plane, this definition implies that the function  $\mathcal{X}f$  is equal to zero on the

points such that  $f(x, y) = 0$ . In the article [53] an invariant curve is defined as a  $\mathcal{C}^1$  function  $f(x, y)$  defined in the open set  $\mathcal{U} \subseteq \mathbb{R}^2$ , such that, the function  $\mathcal{X}f$  is zero in all the points  $\{(x, y) \in \mathcal{U} \mid f(x, y) = 0\}$ . We notice that our definition of invariant curve is a particular case of the previous one but, for the sake of our results, the cofactor is very important and that's why we always assume its existence.

When the cofactor  $k(x, y)$  is a polynomial, we say that  $f(x, y) = 0$  is an invariant curve with polynomial cofactor. We only admit invariant curves with polynomial cofactor of degree lower or equal than  $d - 1$ , that is  $\deg k(x, y) \leq d - 1$ , where  $d$  is the degree of system (7).

The notion of invariant curve is a generalization of the notion of invariant algebraic curve. An *invariant algebraic curve* is an algebraic curve  $f(x, y) = 0$ , where  $f(x, y) \in \mathbb{C}[x, y]$ , which is invariant by the flow of system (7). This condition equals to  $\mathcal{X}f = kf$ , where the cofactor of an invariant algebraic curve is always a polynomial of degree  $\deg k(x, y) \leq d - 1$ .

We cite [78, 99, 100] as compendiums of the results on invariant algebraic curves. For instance, in [78], it is shown that if  $f(x, y) = 0$  and  $g(x, y) = 0$  are two invariant algebraic curves of system (7) with cofactors  $k_f(x, y)$  and  $k_g(x, y)$ , respectively, then the product of the two polynomials gives rise to the curve  $(fg)(x, y) = 0$  which is also an invariant algebraic curve of system (7) and whose cofactor is  $k_f(x, y) + k_g(x, y)$ .

In order to state the known results of integrability using invariant algebraic curves, we need to consider complex algebraic curves  $f(x, y) = 0$ , where  $f(x, y) \in \mathbb{C}[x, y]$ . Since system (7) is defined by real polynomials, if  $f(x, y) = 0$  is an invariant algebraic curve with cofactor  $k(x, y)$ , then its conjugate  $\bar{f}(x, y) = 0$  is also an invariant algebraic curve with cofactor  $\bar{k}(x, y)$ . Hence, its product  $f(x, y)\bar{f}(x, y) \in \mathbb{R}[x, y]$  gives rise to a real invariant algebraic curve with a real cofactor  $k(x, y) + \bar{k}(x, y)$ . For a sake of simplicity, we consider invariant algebraic curves defined by polynomials in  $\mathbb{C}[x, y]$ , although we always keep in mind the previous observation. In  $\mathbb{R}^2$ , the curve given by  $f(x, y) = 0$ , where  $f(x, y)$  is a real function, may only contain a finite number of isolated singular points or be the null set.

An algebraic curve  $f(x, y) = 0$  is called *irreducible* when  $f(x, y)$  is an irreducible polynomial in the ring  $\mathbb{C}[x, y]$ . We can assume, without loss of generality, that  $f(x, y)$  is an irreducible polynomial in  $\mathbb{C}[x, y]$ , because if  $f(x, y)$  is reducible, then all its proper factors give rise to invariant algebraic curves. Given an algebraic curve  $f(x, y) = 0$ , we can always assume that the polynomial  $f(x, y)$  has no multiple factors, that is, its decomposition in the ring  $\mathbb{C}[x, y]$  is of the form  $f(x, y) = f_1(x, y)f_2(x, y) \dots f_\ell(x, y)$ , where  $f_i(x, y)$  are irreducible polynomials and  $f_i(x, y) \neq cf_j(x, y)$  if  $i \neq j$  and for any  $c \in \mathbb{C}$ . The assumption that given an algebraic curve  $f(x, y) = 0$ , the polynomial  $f(x, y)$  has no multiple factors is mainly used to ensure that we do not consider “false” singular points. If  $p$  is a point such that

$f(p) = 0$  and  $\nabla f(p) = 0$ , and  $f(x, y)$  has no multiple factors, then  $p$  is a singular point of the curve  $f(x, y) = 0$ . But, if  $f(x, y)$  has multiple factors, for instance,  $f(x, y) = f_1(x, y)^2$  where  $f_1(x, y)$  is an irreducible polynomial in  $\mathbb{C}[x, y]$ , then all the points of the curve  $\{p \mid f_1(p) = 0\}$  satisfy the property that  $f(p) = 0$  and  $\nabla f(p) = 0$  although they are not all singular points.

We recall that if  $p$  is a singular point of an invariant algebraic curve  $f(x, y) = 0$  of a system (7), then  $p$  is a singular point of the system. Given an algebraic curve  $f(x, y) = 0$ , we will always assume that the decomposition of  $f(x, y)$  in the ring  $\mathbb{C}[x, y]$  has no multiple factors. We want to generalize this property to invariant curves, that's why we will always assume that, given an invariant curve  $f(x, y) = 0$ , if  $p \in \mathcal{U}$  is such that  $f(p) = 0$  and  $\nabla f(p) = 0$ , then  $p$  is a singular point of system (7). This technical hypothesis generalizes the notion of not having multiple factors for algebraic curves. In [23], a set of necessary conditions for a system (7) to have an irreducible invariant algebraic curve is given.

Invariant algebraic curves are the main objects used in the Darboux theory of integrability. In [38], G. Darboux gives a method for finding an explicit first integral for a system (7) in case that  $d(d+1)/2 + 1$  different irreducible invariant algebraic curves are known, where  $d$  is the degree of the system. In this case, a first integral of the form  $H = f_1^{\lambda_1} f_2^{\lambda_2} \dots f_s^{\lambda_s}$ , where each  $f_i(x, y) = 0$  is an invariant algebraic curve for system (7) and  $\lambda_i \in \mathbb{C}$  not all of them null, for  $i = 1, 2, \dots, s$ ,  $s \in \mathbb{N}$ , can be constructed. The functions of this type are called *Darboux functions*.

As we have already stated, given an invariant algebraic curve  $f(x, y) = 0$  whose imaginary part is not null, then its conjugate is also an invariant algebraic curve. Moreover, as system (7) is real, if  $f(x, y)$  appears in the expression of a first integral of the form given by Darboux with exponent  $\lambda$ , then  $\bar{f}(x, y)$  appears in the same expression with exponent  $\bar{\lambda}$ . We call  $\text{Re}f$  the real part of the polynomial  $f$  and by  $\text{Im}f$  its imaginary part. Analogously, let us call  $\text{Re}\lambda$  the real part of the complex number  $\lambda$  and by  $\text{Im}\lambda$  its imaginary part. We call  $\mathbf{i} = \sqrt{-1}$  and we use the following formula for complex numbers:

$$\arctan(z) = \log \left[ \left( \frac{1 - \mathbf{i}z}{1 + \mathbf{i}z} \right)^{i/2} \right], \quad z \in \mathbb{C},$$

to show that

$$\begin{aligned} f^\lambda \bar{f}^{\bar{\lambda}} &= (\text{Re}f + \text{Im}f \mathbf{i})^{\text{Re}\lambda + \text{Im}\lambda \mathbf{i}} (\text{Re}f - \text{Im}f \mathbf{i})^{\text{Re}\lambda - \text{Im}\lambda \mathbf{i}} \\ &= ((\text{Re}f)^2 + (\text{Im}f)^2)^{\text{Re}\lambda} \exp \left\{ -2 \text{Im}\lambda \arctan \left( \frac{\text{Im}f}{\text{Re}f} \right) \right\}. \end{aligned}$$

We deduce that the product  $f(x, y)^\lambda \bar{f}(x, y)^{\bar{\lambda}}$  is a real function and so it is any Darboux function  $H = f_1^{\lambda_1} f_2^{\lambda_2} \dots f_s^{\lambda_s}$ .



We have that the Darboux function  $H$  can be defined in the open set  $\mathbb{R}^2 \setminus \Sigma$ , where  $\Sigma = \{(x, y) \in \mathbb{R}^2 \mid (f_1 \cdot f_2 \cdot \dots \cdot f_r)(x, y) = 0\}$ . We remark that, particularly, if  $\lambda_i \in \mathbb{Z}$ ,  $\forall i = 1, 2, \dots, r$ ,  $H$  is a *rational first integral* for system (7). In this sense J. P. Jouanolou [76], showed that if at least  $d(d+1)+2$  different irreducible invariant algebraic curves are known, then there exists a rational first integral.

The main fact used to prove Darboux's theorem (and Jouanolou's improvement) is that the cofactor corresponding to each invariant algebraic curve is a polynomial of degree  $\leq d-1$ . Invariant curves with polynomial cofactor can also be used in order to find a first integral for the system. This observation enables a generalization of the Darboux's theory which is given in [52], where, for instance, non-algebraic invariant curves with an algebraic cofactor for a polynomial system of degree 4 are presented. In [60], other examples are given of such invariant curves with polynomial cofactor for some families of systems and the way they are used to construct explicit first integrals and inverse integrating factors for the corresponding systems. As a continuation of [60], in [61] we study when a planar differential system polynomial in one variable linearizes in the sense that it has an inverse integrating factor which can be constructed by means of the solutions of linear differential equations and we describe some families of differential systems which are Darboux integrable and whose inverse integrating factor is constructed using the solutions of a second-order linear differential equation defining a family of orthogonal polynomials.

Some generalizations of the classical Darboux theory of integrability may be found in the literature. For instance, independent singular points can be taken into account to reduce the number of invariant algebraic curves necessary to ensure the Darboux integrability of the system, see [29]. A good summary of many of these generalizations can be found in [90] and a survey on the integrability of two-dimensional systems can be found in [21]. One of the most important definitions in this sense is the notion of exponential factor which is given by C. Christopher in [31], when he studies the multiplicity of an invariant algebraic curve. The notion of exponential factor is a particular case of invariant curve for system (7). Given two coprime polynomials  $h, g \in \mathbb{R}[x, y]$ , the function  $e^{h/g}$  is called an *exponential factor* for system (7) if for some polynomial  $k$  of degree at most  $d-1$ , where  $d$  is the degree of the system, the following relation is fulfilled:

$$P \left( \frac{\partial e^{h/g}}{\partial x} \right) + Q \left( \frac{\partial e^{h/g}}{\partial y} \right) = k(x, y) e^{h/g}.$$

As before, we say that  $k(x, y)$  is the *cofactor* of the exponential factor  $e^{h/g}$ .

The next proposition, proved in [31], gives the relationship between the notion of invariant algebraic curve and exponential factor.

**Proposition 2** [31] *If  $F = e^{h/g}$  is an exponential factor and  $g$  is not a constant, then  $g = 0$  is an invariant algebraic curve, and  $h$  satisfies the equation  $P \frac{\partial h}{\partial x} + Q \frac{\partial h}{\partial y} = h k_g + g k_F$  where  $k_g$  and  $k_F$  are the cofactors of  $g$  and  $F$ , respectively.*

The notion of exponential factor is very important in the Darboux theory of integrability since it does not only allow the construction of first integrals following the same method described by Darboux, but it also explains the meaning of the multiplicity of an invariant algebraic curve in relation with the differential system (7). A complete work on this subject can be found in [34].

In the same way as with invariant algebraic curves, given an exponential factor  $F = \exp\{h/g\}$ , since system (7) is a real system, there is no lack of generality in considering that  $h(x, y), g(x, y) \in \mathbb{R}[x, y]$ . If  $F = \exp\{h/g\}$  is an exponential factor with non-null imaginary part, then its complex conjugate,  $\bar{F} = \exp\{\bar{h}/\bar{g}\}$  is also an exponential factor, as it can be easily checked by its defining equation. Moreover, the product  $F \bar{F} = \exp\{h/g + \bar{h}/\bar{g}\}$  is a real exponential factor with a real cofactor.

Since the notion of exponential factor is the most current generalization in the Darboux theory of integrability, any function of the form:

$$f_1^{\lambda_1} f_2^{\lambda_2} \cdots f_r^{\lambda_r} \left( \exp \left( \frac{h_1}{g_1^{n_1}} \right) \right)^{\mu_1} \left( \exp \left( \frac{h_2}{g_2^{n_2}} \right) \right)^{\mu_2} \cdots \left( \exp \left( \frac{h_\ell}{g_\ell^{n_\ell}} \right) \right)^{\mu_\ell}, \quad (9)$$

where  $r, \ell \in \mathbb{N}$ ,  $f_i(x, y) = 0$  ( $1 \leq i \leq r$ ) and  $g_j(x, y) = 0$  ( $1 \leq j \leq \ell$ ) are invariant algebraic curves of system (7),  $h_j(x, y)$  ( $1 \leq j \leq \ell$ ) are polynomials in  $\mathbb{C}[x, y]$ ,  $\lambda_i$  ( $1 \leq i \leq r$ ) and  $\mu_j$  ( $1 \leq j \leq \ell$ ) are complex numbers and  $n_j$  ( $1 \leq j \leq \ell$ ) are non-negative integers, is called a (*generalized*) *Darboux function*.

Let us present a short survey about the Darboux method and its improvements. Let us recall that a singular point  $(x_0, y_0)$  of system (7) is called *weak* if the divergence,  $\text{div} \mathcal{X}$ , of system (7) at  $(x_0, y_0)$  is zero. We recall that  $\mathcal{X}$  denotes the vector field associated to system (7). We denote by  $\mathbb{C}_{d-1}[x, y]$  the set of polynomials in  $\mathbb{C}[x, y]$  of degree lower than  $d$ . We say that  $s$  points  $(x_k, y_k) \in \mathbb{C}^2$ ,  $k = 1, 2, \dots, s$ , are *independent* with respect to  $\mathbb{C}_{d-1}[x, y]$  if the intersection of the  $s$  hyperplanes

$$\left\{ (a_{ij}) \in \mathbb{C}^{d(d+1)/2} : \sum_{i+j=0}^{d-1} x_k^i y_k^j a_{ij} = 0 \right\}_{k=1,2,\dots,s}$$

is a linear subspace of  $\mathbb{C}^{d(d+1)/2}$  of dimension  $d(d+1)/2 - s > 0$ .

The main results about the Darboux method and its improvements are summarized in the following theorem, which can be found in [80], see also [90].

**Theorem 3** *Suppose that a polynomial differential system (7) of degree  $d$  admits  $r$  irreducible invariant algebraic curves  $f_i = 0$  with cofactors  $K_i$  for  $i = 1, 2, \dots, r$ ;  $\ell$  exponential factors  $\exp(h_j/g_j^{n_j})$  with cofactors  $L_j$  for  $j = 1, 2, \dots, \ell$ ; and  $s$  independent singular points  $(x_k, y_k)$  such that  $f_i(x_k, y_k) \neq 0$  for  $i = 1, 2, \dots, r$  and for  $k = 1, 2, \dots, s$ . Moreover, the irreducible factors of the polynomials  $g_j$  are some  $f_i$ 's.*

- (a) *There exist  $\lambda_i, \mu_j \in \mathbb{C}$  not all zero such that  $\sum_{i=1}^r \lambda_i K_i + \sum_{j=1}^{\ell} \mu_j L_j = 0$ , if and only if the (multivalued) function (9) is a first integral of system (7).*
- (b) *If  $r + \ell + s = [d(d+1)/2] + 1$ , then there exist  $\lambda_i, \mu_j \in \mathbb{C}$  not all zero such that  $\sum_{i=1}^r \lambda_i K_i + \sum_{j=1}^{\ell} \mu_j L_j = 0$ .*
- (c) *If  $r + \ell + s \geq [d(d+1)/2] + 2$ , then system (7) has a rational first integral, and consequently all trajectories of the system are contained in invariant algebraic curves.*
- (d) *There exist  $\lambda_i, \mu_j \in \mathbb{C}$  not all zero such that  $\sum_{i=1}^r \lambda_i K_i + \sum_{j=1}^{\ell} \mu_j L_j = \operatorname{div} \mathcal{X}$  if and only if the function (9) is an inverse integrating factor of system (7).*
- (e) *If  $r + \ell + s = d(d+1)/2$  and  $s$  independent singular points are weak, then the function (9) for convenient  $\lambda_i, \mu_j \in \mathbb{C}$  not all zero is a first integral or an inverse integrating factor of system (7).*

Introducing the notion of multiplicity of the invariant algebraic hypersurfaces of a polynomial vector field in  $\mathbb{C}^n$ , the results of Darboux integrability theory of Theorem 3 have been generalized to systems in  $\mathbb{C}^n$ , where  $n \geq 2$ , see [82] and the references therein.

An improvement of the previous Darboux theorem is presented in [18] when the system has a center. As usual  $[q]$  means the integer part of the real number  $q$ .

**Theorem 4** [18] *Consider a polynomial system (7) of degree  $d$ , with a center at the origin and with an arbitrary linear part. Suppose that this system admits  $d(d+1)/2 - [(d+1)/2]$  invariant algebraic curves or exponential factors. Then this system has a Darboux inverse integrating factor.*

In the following section we present several relations between the existence of an inverse integrating factor and the center problem.

We recall that the integrability problem consists in finding the class of functions a first integral of a given system (7) must belong to. We have system (7) defined in a certain class of functions, in this case, the polynomials with real coefficients

$\mathbb{R}[x, y]$ , and we consider the problem whether there is a first integral in another, possibly larger, class. For instance in [92], H. Poincaré stated the problem of determining when a system (7) has a rational first integral. The works of M.J. Prele and M.F. Singer [94] and M.F. Singer [101] go on this direction since they give a characterization of when a polynomial system (7) has an elementary or a Liouvillian first integral. An important fact of their results is that invariant algebraic curves play a distinguished role in this characterization. Moreover, this characterization is expressed in terms of the inverse integrating factor.

Roughly speaking, an *elementary function* is a function constructed from rational functions by using algebraic operations, composition and exponentials, applied a finite number of times, and a *Liouvillian function* is a function constructed from rational functions by using algebraic operations, composition, exponentials and integration, applied a finite number of times. A precise definition of these classes of functions is given in [94, 101]. We are mainly concerned with Liouvillian functions but we will state some results related to integration of a system (7) by means of elementary functions.

We recall that  $\mathbb{C}(x, y)$  denotes the quotient field associated to the ring of polynomials with complex coefficients, that is,  $\mathbb{C}(x, y)$  is the field of rational functions with complex coefficients.

**Theorem 5** [94] *If the system (7) has an elementary first integral, then there exist  $w_0, w_1, \dots, w_n$  algebraic over the field  $\mathbb{C}(x, y)$  and  $c_1, c_2, \dots, c_n$  in  $\mathbb{C}$  such that the elementary function*

$$H = w_0 + \sum_{i=1}^n c_i \ln(w_i) \quad (10)$$

*is a first integral of system (7).*

The existence of an elementary first integral is intimately related to the existence of an algebraic inverse integrating factor, as the following result shows.

**Theorem 6** [94] *If the system (7) has an elementary first integral, then there is an inverse integrating factor of the form*

$$V = \left( \frac{A(x, y)}{B(x, y)} \right)^{1/N},$$

*where  $A, B \in \mathbb{C}[x, y]$  and  $N$  is an integer number.*

The paper [22] is devoted to study which is the form of the inverse integrating factor of a polynomial planar system (7) with a Darboux first integral  $H$  of the form (9). This work is an improvement of the results of Prele and Singer in [94] where it is shown that these Darboux integrable vector fields have a rational inverse integrating factor (see Theorem 7 of [94]). In [22], another proof of this result is presented.

**Theorem 7** [22] *If the system (7) has a (generalized) Darboux first integral of the form (9), then there is a rational inverse integrating factor, that is, an inverse integrating factor of the form:*

$$V = \frac{A(x, y)}{B(x, y)},$$

where  $A, B \in \mathbb{C}[x, y]$ .

Unfortunately, not all the elementary functions of the form (10) are of (generalized) Darboux type. That's why, we can find systems with an elementary first integral and without a rational inverse integrating factor. The following example is of this type. The system appears in the works of Jean Moulin-Ollagnier [85, 86], although he does not give an explicit expression for the first integral. The Lotka-Volterra system:

$$\dot{x} = x \left( 1 - \frac{x}{2} + y \right), \quad \dot{y} = y \left( -3 + \frac{x}{2} - y \right), \quad (11)$$

has the irreducible invariant algebraic curves  $x = 0$ ,  $y = 0$  and  $f(x, y) = 0$ , where  $f(x, y) := (x - 2)^2 - 2xy$ . Applying the results described in [23], it can be shown that this system has no other irreducible invariant algebraic curve. The function  $V(x, y) = x^{-1/2}y^{1/2}f(x, y)$  is the only algebraic inverse integrating factor of system (11) (modulus multiplication by non null constants). Since there is no rational inverse integrating factor, we deduce, by Theorem 7, that there is no (generalized) Darboux first integral. An elementary first integral for this system, which is of the form (10), is given by:

$$H(x, y) := \sqrt{2}\sqrt{x}\sqrt{y} + \ln(x - 2 + \sqrt{2}\sqrt{x}\sqrt{y}) - \ln(x - 2 - \sqrt{2}\sqrt{x}\sqrt{y}).$$

We remark that both Theorems 6 and 7 give a necessary condition to have an elementary or (generalized) Darboux, respectively, first integral. The reciprocals to the statements of Theorems 6 and 7 are not true. A result to clarify the easiest functional class of the first integral once we know the inverse integrating factor appears in [16], see also [49], where the following theorem is stated:

**Theorem 8** [16] *If the system (7) has a rational inverse integrating factor, then the system has a (generalized) Darboux first integral.*

In any case, the following Theorem 9 ensures that given an algebraic inverse integrating factor, there is a Liouvillian first integral. The Liouvillian class of functions contains the rational, algebraic, Darboux and elementary classes of functions.

M.F. Singer shows in [101] the characterization of the existence of a Liouvillian first integral for a system (7) by means of its invariant algebraic curves.

**Theorem 9** [101] *System (7) has a Liouvillian first integral if, and only if, there is an inverse integrating factor of the form  $V = \exp \left\{ \int_{(x_0, y_0)}^{(x, y)} \eta \right\}$ , where  $\eta$  is a rational 1-form such that  $d\eta \equiv 0$ .*

We recall that when 1-form  $\eta$  is such that  $d\eta \equiv 0$ , we say that it is *closed* and if there exists a function  $\varphi$  such that  $\eta = d\varphi$ , we say that  $\eta$  is *exact*.

Taking into account Theorem 9, C. Christopher in [32] gives the following result, which makes precise the form of the inverse integrating factor.

**Theorem 10** [32] *If the system (7) has an inverse integrating factor of the form  $\exp \left\{ \int_{(x_0, y_0)}^{(x, y)} \eta \right\}$ , where  $\eta$  is a rational 1-form such that  $d\eta \equiv 0$ , then there exists an inverse integrating factor of system (7) of the form*

$$V = \exp\{D/E\} \prod C_i^{l_i},$$

where  $D$ ,  $E$  and the  $C_i$  are polynomials in  $x$  and  $y$  and  $l_i \in \mathbb{C}$ .

We notice that  $C_i = 0$  are invariant algebraic curves and  $\exp\{D/E\}$  is an exponential factor for system (7). In fact, since system (7) is a real system, we can assume, without loss of generality, that  $V$  is a real function.

Theorem 10 states that the search for Liouvillian first integrals can be reduced to the search of invariant algebraic curves and exponential factors. Therefore, if we characterize the possible cofactors, we have the invariant algebraic curves of a system and, hence, its Liouvillian or non Liouvillian integrability.

Several works study the relation between the existence of invariant algebraic curves and the integrability of the system. The existence of an inverse integrating factor and the functional class it belongs to is crucial in the resolution of the integrability problem, as Theorems 7, 6 and 10 show. A number high enough of invariant algebraic curves of system (7) implies its integrability in one of the rational, elementary or Liouvillian class, due to Darboux's theorem and Jouanolou's improvement, see also Theorems 16 and 17. The degree of an invariant algebraic curve is not necessarily related with the integrability class of the system, see [28, 85] and the references therein.

We conclude this part with a theorem that summarizes some relations between inverse integrating factors and first integrals of polynomial vector fields.

**Theorem 11** *Let  $\mathcal{X}$  be a planar polynomial vector field.*

- (i) *If  $\mathcal{X}$  has a Liouvillian first integral, then it has a Darboux inverse integrating factor.*

- (ii) If  $\mathcal{X}$  has a Darboux first integral, then it has a rational inverse integrating factor.
- (iii) If  $\mathcal{X}$  has a polynomial first integral then it has a polynomial inverse integrating factor.

Statement (i) of Theorem 11 was proved in [101] and [32] and statements (ii) and (iii) in [22].

Another problem related with the inverse integrating factor and the integrability problem is an inverse problem: given a function  $V(x, y)$ , the question is to find (all the) planar differential systems with  $V(x, y)$  as inverse integrating factor. In the case of searching for a Darboux inverse integrating factor, a very exhaustive approach to this problem is given in [33, 80, 90]. The main result of [33] establishes, under two generic conditions, **all** the planar polynomial differential systems with an inverse integrating factor of the form  $V(x, y) = f_1^{\lambda_1} f_2^{\lambda_2} \dots f_s^{\lambda_s}$ , where each  $f_i(x, y) = 0$  is an invariant algebraic curve of the system and  $\lambda_i \in \mathbb{C}$ , for  $i = 1, 2, \dots, s$ ,  $s \in \mathbb{N}$ . We do not reproduce the main result of [33] because a lot of notation would need to be introduced.

In [17] another method to construct systems with a given inverse integrating factor is described. In fact, in 1997 the function  $V$  was named *null divergence factor*. This method is a generalization of the classical Darboux method to generate integrable systems. One of the main results in this paper is the following one.

**Theorem 12** [17] *Let  $\mathcal{X}_i = P_i(x, y)\partial_x + Q_i(x, y)\partial_y$ , with  $i = 1, 2, \dots, n$ , be  $\mathcal{C}^1$  vector fields defined in an open subset  $\mathcal{U} \subseteq \mathbb{R}^2$ , which have  $\mathcal{C}^2$  inverse integrating factors  $V_i(x, y)$ , respectively. Then, the vector field  $\mathcal{X} = P(x, y)\partial_x + Q(x, y)\partial_y$  with*

$$\begin{aligned} P &= \lambda_0 \frac{\partial V}{\partial y} + \sum_{i=1}^n \lambda_i \left( \prod_{j=1, j \neq i}^n V_j \right) P_i, \\ Q &= -\lambda_0 \frac{\partial V}{\partial x} + \sum_{i=1}^n \lambda_i \left( \prod_{j=1, j \neq i}^n V_j \right) Q_i, \end{aligned}$$

where  $\lambda_i$  are arbitrary real numbers for  $i = 0, 1, 2, \dots, n$ , has the inverse integrating factor  $V(x, y)$  given by  $V(x, y) = \prod_{i=1}^n V_i(x, y)$ .

Indeed, if two systems have the same inverse integrating factor, a more general system which has such inverse integrating factor can be constructed, as it is shown in the following proposition.

**Proposition 13** [17] *Let  $\mathcal{X}_i = P_i(x, y)\partial_x + Q_i(x, y)\partial_y$  with  $i = 1, 2$ , be two  $C^1$  vector fields defined in an open subset  $\mathcal{U} \subseteq \mathbb{R}^2$ , which have the same inverse integrating factor  $V(x, y)$ . Then, the vector field  $\mathcal{X}_1 + \lambda\mathcal{X}_2$  has also the function  $V(x, y)$  as an inverse integrating factor, for arbitrary values of the real parameter  $\lambda$ .*

This proposition establishes that the set of vector fields with the same inverse integrating factor forms a  $\mathbb{R}$  vector space.

A polynomial inverse integrating factor allows the study of the dynamics of system (7), because a first integral can be computed, but it is not so involving as looking for a polynomial first integral. Indeed, once the degree of a polynomial inverse integrating factor is fixed, by an ansatz for instance, the problem of looking for it is reduced to a system of linear equations on its coefficients. Many authors have used this idea to find families of planar polynomial differential systems of the form (7) for which all the dynamics can be determined through an inverse integrating factor.

In [20], necessary conditions for a planar polynomial vector field to have a polynomial inverse integrating factor are obtained, see also [49]. All the quadratic systems with a polynomial inverse integrating factor are determined in [36] and all the quadratic systems with a polynomial first integral are given in [14].

In [9] all polynomial first integrals of the non-homogeneous two-dimensional Lotka–Volterra system of ordinary differential equations are determined and the role of polynomial inverse integrating factors is emphasized. Indeed, new first integrals of this class of systems having a polynomial inverse integrating factor is presented. The Liouvillian integrability of Lotka–Volterra systems has been studied in [86, 8].

In the work [13], planar differential systems of the form (7) and defined by the sum of homogeneous vector fields are studied. In particular systems with degenerate infinity are taken into account. Let us denote by  $P_d(x, y)$  and  $Q_d(x, y)$  the terms of the highest degree  $d$  in system (7). We say that system (7) is of *degenerate infinity* if  $xQ_d(x, y) - yP_d(x, y) \equiv 0$ . We remark that when a system (7) with degenerate infinity is embedded into a compact space (either by the Poincaré compactification into an sphere or when it is embedded in the complex projective plane) the line at infinity is filled with singular points.

We recall that a real function  $H(x, y)$  is said to be  $p$ -degree homogeneous if  $H(\lambda x, \lambda y) = \lambda^p H(x, y)$  for all  $(x, y)$  in the domain of definition of  $H(x, y)$  and for all  $\lambda \in \mathbb{R}$ , where  $p \in \mathbb{Z}$ .

One of the main results in [13] is the following one.



**Theorem 14** [13] *Let us consider the following planar polynomial differential system*

$$\dot{x} = P_n(x, y) + x A_{d-1}(x, y), \quad \dot{y} = Q_n(x, y) + y A_{d-1}(x, y), \quad (12)$$

where  $P_n(x, y)$  and  $Q_n(x, y)$  are homogeneous real polynomials of degree  $n$ ,  $A_{d-1}(x, y)$  is a real homogeneous polynomial of degree  $d - 1$  and  $d > n \geq 1$ . Let us also consider the related homogeneous polynomial differential system:

$$\dot{x} = P_n(x, y), \quad \dot{y} = Q_n(x, y). \quad (13)$$

Then, the following statements hold.

- (a) *If  $H(x, y)$  is a  $p$ -degree homogeneous first integral of system (13), then  $H(x, y)$  is a particular solution of system (12).*
- (b) *The homogeneous function  $V_{n+1}(x, y) := xQ_n(x, y) - yP_n(x, y)$  is an inverse integrating factor of system (13).*
- (c) *The homogeneous function  $V_{n+1}(x, y) := xQ_n(x, y) - yP_n(x, y)$  is a particular solution of system (12).*
- (d) *If  $H(x, y)$  is a  $p$ -degree homogeneous first integral of system (13). Then, the function*

$$(xQ_n(x, y) - yP_n(x, y)) H(x, y)^{\frac{d-n}{p}}$$

*is a (generalized) Darboux inverse integrating factor of system (12).*

The degree of a polynomial inverse integrating factor in relation with the degree  $d$  of the system can be bounded under certain conditions. The conditions established in the following result come from the embedding of a planar vector field in  $\mathbb{CP}^2$ , see [23] for the complete definition of this embedding. Consider the polynomial differential system (7) with  $P$  and  $Q$  coprime polynomials of maximum degree  $d$ . Extending system (7) to a differential equation in the complex projective plane  $\mathbb{CP}^2$ , a point  $(X_0 : Y_0 : 0) \in \mathbb{CP}^2$  is termed *infinite singular point* of system (7) if  $(X_0, Y_0) \in \mathbb{C}^2$  is a root of the homogeneous polynomial  $yP_d(x, y) - xQ_d(x, y)$ . Here  $P_d$  and  $Q_d$  denote the highest homogeneous components of  $P$  and  $Q$  of degree  $d$ . Following Seidenberg, a singular point is called *simple* if the eigenvalues  $\lambda, \mu \in \mathbb{C}$  associated to its linear part satisfy  $\lambda \neq \mu \neq 0$  and  $\lambda/\mu \notin \mathbb{Q}^+$ , where  $\mathbb{Q}^+$  stands for the positive rational numbers. Given a polynomial  $F(x, y)$  of degree  $n$ , we denote by  $\tilde{F}(X, Y, Z)$  its projectivization, that is, the homogeneous polynomial  $\tilde{F}(X, Y, Z) = Z^n F(X/Z, Y/Z)$ . Taking into account these definitions we can state the following result of Walcher proved in [104].

**Theorem 15** [104] *Let  $V(x, y)$  be a polynomial inverse integrating factor of a polynomial system  $\dot{x} = P(x, y)$ ,  $\dot{y} = Q(x, y)$  of degree  $d$  with  $P$  and  $Q$  coprime. Assume that the highest homogeneous components of  $P$  and  $Q$  of degree  $d$  are coprime. If there is an infinite simple critical point of the system, then the degree of  $V$  is exactly  $d + 1$ .*

We observe that Theorem 15 is also proved in [12] with the additional assumption that the singularity at infinity  $p$  satisfies  $\tilde{V}(p) = 0$  where  $\tilde{V}$  is the projectivization of  $V$ . We remark that the approach introduced in [104] uses analytical techniques such as the Poincaré–Dulac normal form and the proof given in [12] is completely algebraic and based on the extension of differential equations to the complex projective plane and the results of Seidenberg about the reduction of singularities. The structure of polynomial inverse integrating factors is also studied by Walcher in [105, 106].

In relation with rational first integrals and in order to state the main result of [22], we need to introduce some preliminary concepts, see also [48]. Let  $H = f/g$  be a rational first integral of a polynomial system (7). According to Poincaré [92] we say that  $c \in \mathbb{C} \cup \{\infty\}$  is a *remarkable value* of  $H$  if  $f + cg$  is a reducible polynomial in  $\mathbb{C}[x, y]$  (here,  $c = \infty$  means that  $f + cg$  denotes just  $g$ ). In the work [22] it is proved that there are finitely many remarkable values for a given rational first integral  $H$ .

Let now  $H$  be a polynomial first integral of degree  $n$  of a polynomial system (7). We say that the degree of  $H$  is *minimal* between all the degrees of the polynomial first integrals of (7) if any other polynomial first integral of (7) has degree  $\geq n$ .

Assume  $H = f/g$  to be a rational first integral. Hence, we say that  $H$  has *degree  $n$*  if  $n$  is the maximum of the degrees of  $f$  and  $g$ . Moreover, we say that the degree of  $H$  is minimal between all the degrees of the rational first integrals of system (7) if any other rational first integral of (7) has degree  $\geq n$ .

Now suppose that  $c \in \mathbb{C}$  is a remarkable value of a rational first integral  $H = f/g$  and that  $\prod_{i=1}^r u_i^{\alpha_i}$  is the factorization of the polynomial  $f + cg$  into irreducible factors in  $\mathbb{C}[x, y]$ . If some of the  $\alpha_i$  is larger than 1, then we say that  $c$  is a *critical remarkable value* of  $H$  and that  $u_i = 0$  having  $\alpha_i > 1$  is a critical remarkable invariant algebraic curve of (7) with exponent  $\alpha_i$ .

Finally, let  $f$  be a polynomial. We denote by  $\tilde{f}$  the homogeneous part of  $f$  of highest degree and this notation is also used for a Darboux functions like (9).

The main result of [22] is the following one.

**Theorem 16** [22] *Suppose that a complex polynomial vector field  $\mathcal{X} = P(x, y)\partial_x + Q(x, y)\partial_y$  of degree  $d$  with  $P$  and  $Q$  coprime has a Darboux first integral  $H$  given by (9) where the polynomials  $f_i$  and  $g_i$  are irreducible and the polynomials  $g_i$  and  $h_i$  are coprime in  $\mathbb{C}[x, y]$ . Then the following statements hold.*

- (a) *The inverse integrating factor  $V_{\log H}$  associated to the first integral  $\log H$  is a rational function, and it can be written in the form  $V_{\log H} = \prod_{i=1}^m u_i^{k_i}(x, y)$  with  $u_i \in \mathbb{C}[x, y]$  irreducible and  $k_i \in \mathbb{Z}$ . Moreover, if  $\mathcal{X}$  has no rational first integrals, then  $V_{\log H}$  is the unique rational inverse integrating factor of  $\mathcal{X}$ .*
- (b) *Assume that  $H$  is a minimal polynomial first integral. Then there exists a polynomial inverse integrating factor.*
- (c) *Suppose that  $H = f/g$  is a minimal rational first integral of  $\mathcal{X}$  and that  $\mathcal{X}$  has no polynomial first integrals. It is not restrictive to assume that  $f$  and  $g$  are irreducible. Then,*
  - (c.1) *the rational function*

$$V_{f/g} = \frac{g^2}{\prod_i u_i^{\alpha_i - 1}}$$

*where the product runs over all critical remarkable invariant algebraic curves  $u_i = 0$  having exponent  $\alpha_i$  is an inverse integrating factor; and*

- (c.2)  *$\mathcal{X}$  has a polynomial inverse integrating factor if and only if  $H$  has at most two critical remarkable values.*

*Additionally, if we assume for the first integral (9) that  $f_i$  (respectively  $g_j$ ) are different for  $i = 1, \dots, r$  (respectively  $j = 1, \dots, \ell$ ), and that it is complete (i.e. the unique algebraic invariant curves of system  $\mathcal{X}$  are the  $f_i = 0$  and the  $g_j = 0$ ), then the following two statements hold.*

- (d) *If  $\mathcal{X}$  has no rational first integrals, then the inverse integrating factor  $V_{\log H}$  associated to the first integral  $\log H$  is the polynomial*

$$V_{\log H} = \prod_{i=1}^r f_i \prod_{j=1}^{\ell} g_j^{n_j+1}.$$

- (e) *If  $\tilde{H}$  is a multi-valued function and  $\exp(h_j/g_j)$  are exponential factors of  $\mathcal{X}$  for  $j = 1, \dots, \ell$ , then  $V_{\log H} = \prod_{i=1}^r f_i \prod_{j=1}^{\ell} g_j^{n_j+1}$  is a polynomial of degree  $d + 1$ .*

In the particular case that  $\mu_i = 0$  for  $i = 1, \dots, \ell$  in the expression of (9), statement (d) of Theorem 16 can be thought as a generalization of following result due to Kooij and Christopher [77] and independently to Żoładek [109].

**Theorem 17** [77, 109] *Consider a polynomial vector field  $\mathcal{X} = P(x, y)\partial_x + Q(x, y)\partial_y$  of degree  $d$  with  $P, Q \in \mathbb{C}[x, y]$  (resp.  $P, Q \in \mathbb{R}[x, y]$ ) having  $q$  invariant algebraic curves  $f_i = 0$  such that the polynomials  $f_i$  are irreducible in  $\mathbb{C}[x, y]$  (resp.  $\mathbb{R}[x, y]$ )*

and satisfy that no more than two curves meet at any point of the plane  $\mathbb{C}^2$  (resp.  $\mathbb{R}^2$ ) and are not tangent at these points, no two curves have a common factor in their highest order terms and the sum of the degrees of the curves is  $d + 1$ . Then,  $\prod_{i=1}^r f_i$  is an inverse integrating factor of  $\mathcal{X}$  and  $\prod_{i=1}^r f_i^{\lambda_i}$  for convenient  $\lambda_i \in \mathbb{C}$  (resp.  $\lambda_i \in \mathbb{R}$ ) is a first integral of  $\mathcal{X}$ .

From statement (d) of Theorem 16, the following result easily follows.

**Corollary 18** [22] *Suppose that a real polynomial vector field  $\mathcal{X} = P(x, y)\partial_x + Q(x, y)\partial_y$  of degree  $d$  with  $P$  and  $Q$  coprime has a Darboux first integral  $H$  given by (9) where the polynomials  $f_i$  and  $g_j$  are irreducible,  $f_i \neq g_j$ , the polynomials  $g_j$  and  $h_j$  are coprime in  $\mathbb{R}[x, y]$ ,  $\exp(h_j/g_j^{n_j})$  are exponential factors of  $\mathcal{X}$ , the  $\lambda_i$  and  $\mu_j$  are either real numbers, or if some of them is complex then it appears its conjugate. If  $H$  is complete and  $\tilde{H}$  is multi-valued, then  $V_{\log H} = \prod_{i=1}^r f_i \prod_{j=1}^\ell g_j^{n_j+1}$  is a polynomial of degree  $d + 1$ . If the system has foci or limit cycles, these are contained in the set  $\{V_{\log H} = 0\}$ .*

Using Corollary 18 particularized to quadratic ( $d = 2$ ) polynomial vector fields, in [22] it is obtained the next result.

**Corollary 19** [22] *Under the assumptions of Corollary 18 there are no real quadratic polynomial vector fields with a Darboux first integral (9) and a limit cycle.*

Some examples of polynomial systems satisfying the assumptions of Corollary 18 are the following ones, see again [22]:

- $\dot{x} = -y - xf_1(x, y)$ ,  $\dot{y} = x - yf_1(x, y)$  where  $f_1(x, y) = x^2 + y^2 - 1 = 0$  is an invariant circle which becomes an algebraic limit cycle. The origin is a focus and  $f_2(x, y) = x^2 + y^2$  is another invariant algebraic curve. The system possesses the inverse integrating factor  $V = f_1 f_2$ .
- The system  $\dot{x} = y - 4xy$ ,  $\dot{y} = -x + x^2 + 2xy - y^2$  has the invariant algebraic curves  $f_1(x, y) = 1 - 4x$ ,  $f_2(x, y) = \sqrt{2}y + (x + y - 1)i$  and  $f_3(x, y) = \sqrt{2}y - (x + y - 1)i$  with  $i^2 = -1$ . The function  $V(x, y) = f_1 f_2 f_3$  is an inverse integrating factor. Notice that the system has a center at  $(0, 0)$  and a unstable focus at  $(1, 0)$

The recent work [48] is also devoted to study the properties of remarkable values. The polynomial  $R(x, y) := \prod_i u_i^{\alpha_i - 1}$  defined as the product of all remarkable curves powered to their respective exponent minus one, is called the *remarkable factor*. From Theorem 16, if  $H$  is a polynomial first integral, then the remarkable factor  $R$  is a polynomial integrating factor of  $\mathcal{X}$ . Moreover  $R$  divides the product  $\prod(H + c_i)$  where  $c_i$  are all the critical remarkable values of  $H$ . Thus the polynomial

$V_R = \prod(H + c_i)/R$  is an inverse integrating factor of the system. The following theorem gives some relations between the degree of a system with a polynomial first integral, the degree of its inverse integrating factor  $V_R(x, y)$  and the number of critical remarkable values.

**Theorem 20** [48] *Let  $\mathcal{X}$  be a polynomial vector field of degree  $d$  and let  $H$  be a minimal polynomial first integral of  $\mathcal{X}$ . Consider the remarkable factor  $R$  and the polynomial inverse integrating factor  $V_R$ . Let  $k$  be the number of critical remarkable values. Then,*

- (a)  $k(k + d) \leq \deg V_R = k \deg H - \deg R \leq k(\deg H - 1) \leq \deg R(\deg R + d)$   
and
- (b)  $\deg V_R < \deg H$  if and only if  $k = 1$ . Moreover in this case  $\deg V_R = d + 1$ .

## 5 On the center problem

One of the classical problems in the qualitative theory of planar analytic differential systems is to characterize the local phase portrait near an isolated singular point. By using the blow-up technique, see [43], this problem can be solved except when the singularity is monodromic, that is, it is either a focus or a center. The problem of distinguishing between a center or a focus is called the *center problem*. Another interesting problem is to know whether there exists or not a local analytic first integral defined in a neighborhood of a singular point. These two problems are equivalent when the singularity has associated nonvanishing complex conjugated eigenvalues. In this case, translating the singular point at the origin, after a linear change of variables and a rescaling of the time variable, the system can be written into the form:

$$\dot{x} = -y + f(x, y), \quad \dot{y} = x + g(x, y), \quad (14)$$

where  $f(x, y)$  and  $g(x, y)$  are analytic functions near the origin without constant nor linear terms. It is well known since Poincaré and Liapunov that system (14) has a center at the origin if and only if there exists a local analytic first integral of the form  $H(x, y) = x^2 + y^2 + F(x, y)$  defined in a neighborhood of the origin, where  $F$  starts with terms of order higher than 2. We recall here that the *Poincaré–Liapunov constants* are the values  $V_{2k}$  defined from the formal power series  $H(x, y) = \sum_{n=2}^{\infty} H_n(x, y)$ , where  $H_2(x, y) = (x^2 + y^2)/2$  and  $H_n(x, y)$  are homogeneous polynomials of degree  $n$  satisfying  $\mathcal{X}H = \sum_{k=2}^{\infty} V_{2k}(x^2 + y^2)^k$ . The origin is a center of (14) if and only if all the Poincaré–Liapunov constants vanish. When  $V_{2j} = 0$  for  $j = 2, 3, \dots, k-1$  and  $V_{2k} \neq 0$ , we say that the origin of system (14) is a focus of order  $k$ .

The existence of invariant algebraic curves is strongly related with the origin of system (14) being a center, as it is explained in [31, 99, 100].

The proof of the following result is a particular case of a theorem that was given by Reeb in [95] (see also Mattei and Moussu [84] and Moussu [87]). For a proof using elementary methods see [37].

**Theorem 21** [95] *System (14) has a center at the origin if and only if there is a nonzero analytic inverse integrating factor in a neighborhood of the origin.*

In fact, given a system (14), the computational problems of looking for a first integral  $H(x, y) = x^2 + y^2 + \dots$  or for an inverse integrating factor  $V(x, y) = 1 + \dots$ , where the dots denote terms of higher order, are of the same difficulty. Thus, the inverse integrating factor offers an alternative to the solution of the center problem. In [21], it has been noticed that for many systems of type (14) having a center at the origin there is an inverse integrating factor  $V$  with very simple properties which can be globally defined in all  $\mathbb{R}^2$  and which is usually a polynomial. By contrary, the first integral is, in general, a complicated expression that can not be written in terms of elementary functions.

In particular, when in system (14) the functions  $f$  and  $g$  are both quadratic then there exists a polynomial inverse integrating factor of degree 3 or 5, see [10], whereas the first integrals are far more complicated, see [83]. When the functions  $f$  and  $g$  in system (14) are both cubic homogeneous polynomials and the origin is a center, there exists a polynomial inverse integrating factor of degree at most 10, as it is also shown in [10]. In [66], the authors study cubic systems of the form (14) and give some sufficient conditions for the origin to be a center. This conditions come from the imposition to the system to have an inverse integrating factor.

The work [30] is a survey on isochronous centers, that is, centers of the form (14) such that all the periodic orbits surrounding the origin have the same period. Many families of isochronous systems are listed and an explicit expression of an inverse integrating factor is given in each case. We include here a couple of results of the ones appearing in [30] which we have chosen for being the most known examples of isochronous centers in the literature.

The quadratic systems with a isochronous center at the origin are characterized in the following result.

**Theorem 22** *The origin is an isochronous center of a quadratic system (14) if, and only if, the system can be brought by means of an affine change of coordinates and a rescaling of time, to one of the following four systems. For each case in the list we include the corresponding inverse integrating factor  $V(x, y)$ .*

$$(S_1) \quad \dot{x} = -y + x^2 - y^2, \dot{y} = x(1 + 2y), \text{ with } V(x, y) = (1 + 2y)^2.$$

$$(S_2) \quad \dot{x} = -y + x^2, \dot{y} = x(1 + y), \text{ with } V(x, y) = (1 + y)^3.$$

$$(S_3) \quad \dot{x} = -y - \frac{4}{3}x^2, \dot{y} = x(1 - \frac{16}{3}y), \text{ with } V(x, y) = (3 - 16y)(9 - 24y + 32x^2).$$

$$(S_4) \quad \dot{x} = -y + \frac{16}{3}x^2 - \frac{4}{3}y^2, \dot{y} = x(1 + \frac{8}{3}y), \text{ with } V(x, y) = (3 + 8y)(9 + 96y - 256x^2 + 128y^2).$$

Let us consider a cubic polynomial system of the form (14) and let us assume that it contains no quadratic terms, that is, it is the sum of a linear system and a cubic homogeneous system. We say that such a system is cubic and with homogeneous nonlinearities. The following results characterizes which of these systems have an isochronous center at the origin.

**Theorem 23** *The origin is an isochronous center of a cubic system with homogeneous nonlinearities of the form (14) if, and only if, the system can be brought by means of an affine change of coordinates and a rescaling of time, to one of the following four systems. For each case in the list we include the corresponding inverse integrating factor  $V(x, y)$ .*

$$(S_1^*) \quad \dot{x} = -y + x^3 - 3xy^2, \dot{y} = x + 3x^2y - y^3, \text{ with } V(x, y) = (x^2 + y^2)^3.$$

$$(S_2^*) \quad \dot{x} = -y + x^3 - xy^2, \dot{y} = x + x^2y - y^3, \text{ with } V(x, y) = (1 + 2xy)^2.$$

$$(S_3^*) \quad \dot{x} = -y + 3x^2y, \dot{y} = x - 2x^3 + 9xy^2, \text{ with } V(x, y) = (1 - 3x^2)^4.$$

$$(S_4^*) \quad \dot{x} = -y - 3x^2y, \dot{y} = x + 2x^3 - 9xy^2, \text{ with } V(x, y) = (1 + 3x^2)^4.$$

In the works [10, 11], Chavarriga writes system (14) in polar coordinates and studies the existence of inverse integrating factors polynomial in the radial variable. In [24, 25, 26], the authors look for possible inverse integrating factors for polynomial vector fields of the form

$$\mathcal{X} = -y\partial_x + x\partial_y + \mathcal{X}_s, \quad (15)$$

where  $\mathcal{X}_s$  is a polynomial homogeneous vector field of degree  $s \geq 2$ . In particular, they use the quasi-polar coordinates  $(R, \varphi)$  where  $R = r^{s-1}$  and  $(r, \varphi)$  are the polar coordinates, that is,  $x = r \cos \varphi$ ,  $y = r \sin \varphi$ . Next, it is assumed the existence of an inverse integrating factor  $V(R, \varphi)$  of (15) which is polynomial in the variable  $R$ , that is, of the form  $V(R, \varphi) = \sum_{i=0}^p V_i(\varphi)R^i$  with  $V_0(\varphi) \equiv 1$  and where  $V_i(\varphi)$  are homogeneous trigonometrical polynomials of degree  $i(s-1)$ . This assumption is clearly equivalent to impose an inverse integrating factor of the form  $V(R, \varphi) = \prod_{i=1}^p (1 + x_i(\varphi)R)^{\alpha_i}$ , with  $\alpha_i \in \mathbb{R}$ . The authors try to solve the system of equations for the unknown functions  $x_i(\varphi)$  in the cases  $p = 1, 2, 3$ . The case  $p = 1$  is totally solved. If  $p = 2$ , only is solved the case  $\alpha_1 = \alpha_2 = (s+1)/(s-1) \pm 1/2$  with

arbitrary  $s$ . Finally, when  $p = 3$  the following two particular cases are investigated:  $\alpha_1 = \alpha_2 = \alpha_3 = 5/3$  and either  $s = 2$  or  $s = 3$ .

In [27], some invariants are determined from which a formal first integral for system (15) can be computed. Moreover, this technique is applied to the problem of determining the centers of polynomial vector fields (15). Recall that a complete classification of such centers is known when  $s = 2, 3$  but only partial results are known in the cases  $s = 4$  and  $s = 5$ .

Theorem 21 is used in [69] to find conditions to have a center. In this work, Giné proposes a formal power series  $V(x, y) = \sum_{n=0}^{\infty} \bar{V}_n(x, y)$ , where  $\bar{V}_0(x, y) \equiv 1$  and  $\bar{V}_n(x, y)$  are homogeneous polynomials of degree  $n$  such that  $\mathcal{X}V - V\operatorname{div}\mathcal{X} = \sum_{k=2}^{\infty} v_{2k}(x^2 + y^2)^k$ , where  $\mathcal{X}$  is given by (15) and the constants  $v_{2k}$  are called the *inverse integrating factor constants*. Using the above mentioned quasi-polar coordinates  $(R, \varphi)$  it is shown that, if the Poincaré–Liapunov constants  $V_k = 0$  for  $k = 1, \dots, m$  and  $V_{m+1} \neq 0$ , then  $v_{m+1} = -((m+1)(s-1) + 2)V_{m+1}$ . In this spirit, the paper [42] is concerned with the existence of a formal integrating factor of planar analytic system having a non degenerate focus or center at the origin and gives an algorithm to calculate the Poincaré–Liapunov constants of any order.

Given a real analytic planar vector field  $\mathcal{X}_0$  with a center at  $p_0 \in \mathbb{R}^2$ , in [62] the authors say that this center is *limit of a linear type center* if there exists a 1-parameter family  $\mathcal{X}_\epsilon$  of analytic planar vector fields with  $\epsilon \geq 0$ , defined in a neighborhood of  $p_\epsilon \in \mathbb{R}^2$  and having a non degenerate center at  $p_\epsilon$  for all  $\epsilon > 0$  sufficiently small. The main results of [62] are summarized as follows.

**Theorem 24** [62] *Let  $\mathcal{X}_0$  be a real analytic planar vector field with a center at  $p_0 \in \mathbb{R}^2$ . Then, the following holds:*

- (i) *If  $p_0$  is a nilpotent center, then it is limit of a linear type center.*
- (ii) *If  $p_0$  is a Hamiltonian degenerate center, then it is limit of a linear type Hamiltonian center.*
- (iii) *If  $p_0$  is a time-reversible degenerate center, then it is limit of a time-reversible linear type center.*

In the work [70], Giné continues the study of the analytic centers which are limit of linear type centers. It is proved that if a degenerate center has an analytic inverse integrating factor  $V(x, y)$  which does not vanish near the center, then this degenerate center is also the limit of a linear type center (changing the time variable). The idea is as follows. Assume  $V(x, y)$  is an analytic inverse integrating factor of the analytic vector field  $\mathcal{X}_0 = P(x, y)\partial_x + Q(x, y)\partial_y$  such that  $(0, 0)$  is a degenerate center and  $V(0, 0) \neq 0$ . Hence, the rescaled vector field  $\mathcal{X}_0/V$  is



hamiltonian near the origin with analytic first integral  $H(x, y)$ . Therefore, since the perturbed vector field  $\mathcal{X}_\epsilon = \mathcal{X}_0/V + \epsilon(-y\partial_x + x\partial_y)$  is Hamiltonian too, the origin becomes a linear type center of  $\mathcal{X}_\epsilon$  for all  $\epsilon \neq 0$ .

## 6 Limit cycles

Let  $V(x, y)$  be an inverse integrating factor in the open set  $\mathcal{U} \subset \mathbb{R}^2$  of a  $\mathcal{C}^1(\mathcal{U})$  planar vector field  $\mathcal{X}$ . That is, the vector field  $\mathcal{X}/V$  has zero divergence, where defined. If  $\mathcal{W}$  is any simply connected component of  $\mathcal{U} \setminus V^{-1}(0)$ , then the condition  $\text{div}(\mathcal{X}/V) \equiv 0$  implies that  $\mathcal{X}/V$  is *Hamiltonian* on  $\mathcal{W}$  with  $\mathcal{C}^2$  single-valued hamiltonian function  $H : \mathcal{W} \rightarrow \mathbb{R}$ . Since Hamiltonian systems are area-preserving, hence have no limit cycles, and  $\mathcal{X}$  and  $\mathcal{X}/V$  are topologically equivalent, it follows immediately that, in the presence of an inverse integrating factor  $V$ , any limit cycle of  $\mathcal{X}$  lies either in  $V^{-1}(0)$  or in a component of  $\mathcal{U} \setminus V^{-1}(0)$  that is not simply connected. Using the machinery of de Rham cohomology, Giacomini, Llibre, and Viano eliminated the latter possibility in [63]. Hence, they prove the following theorem.

**Theorem 25** [63] *Let  $\gamma$  be a limit cycle of a  $\mathcal{C}^1$  real planar vector field  $\mathcal{X}$  and let  $V$  be any inverse integrating factor of  $\mathcal{X}$  defined in some neighborhood of  $\gamma$ . Then,  $\gamma \subset V^{-1}(0)$ .*

A different proof of Theorem 25 can be found in [4, 56]. We would like to recall here that M.V. Dolov in [39] studies the existence of a single valued regular integrating factor in a neighborhood of a limit cycle and presents some connections between an integrating factor and a limit cycle. Moreover, in the works [40, 41] of Dolov and coauthors, published before the proof of Theorem 25, it is shown that vector fields with a Darboux inverse integrating factor of the form  $V = \exp(R)$  with rational  $R$  cannot have limit cycles.

Theorem 25 has been applied in many papers to study limit cycles of a system as we will see in forthcoming sections. As an example where this theorem is applied, we would like to recall the result of Llibre and Rodríguez in [81] where it is shown that every finite configuration of disjoint simple closed curves of the plane is topologically realizable as the set of limit cycles of a polynomial vector field. Moreover, the realization can be made by algebraic limit cycles, and an explicit polynomial vector field exhibiting any given finite configuration of limit cycles is provided. The proof of this realization makes use of the inverse integrating factor and, in particular, of Theorem 25. A generalization of the result of Llibre and Rodríguez is given in [91] for systems in higher dimension, that is, it is shown that

any finite configuration of (smooth) cycles in  $\mathbb{R}^n$  can be realized (up to global diffeomorphism) as hyperbolic and asymptotically stable limit cycles of a polynomial vector field.

From Theorem 25 we have that the knowledge of an inverse integrating factor for a planar differential system (3) implies the knowledge of the number (and location) of the limit cycles of the system. Many authors have treated the problem of the existence of an inverse integrating factor. For a polynomial system (thus defined in the whole  $\mathbb{R}^2$ ), the knowledge of a polynomial inverse integrating factor solves the question of the number and location of limit cycles of the polynomial system, see Section 4. In [55], the authors study the problem of existence of a polynomial inverse integrating factor in several cases of quadratic vector fields  $\mathcal{X}$ . If such an integrating factor  $V(x, y)$  exists, then from Theorem 25 the curve  $V = 0$  is invariant for  $\mathcal{X}$  and any limit cycle of  $\mathcal{X}$  lies in this curve. Therefore, in [55], the authors study planar quadratic polynomial vector fields that can have limit cycles and study the nonexistence of invariant algebraic curves, polynomial inverse integrating factors and algebraic limit cycles of arbitrary degree for these systems. Ye Yian-Qian [108] classified real quadratic systems that can have limit cycles in the following three families

$$\dot{x} = \delta x - y + \ell x^2 + Mxy + Ny^2, \quad \dot{y} = x(1 + ax + by),$$

according to: family (I) if  $a = b = 0$ ; family (II) if  $a \neq 0$  and  $b = 0$ ; family (III) if  $b \neq 0$ . In [55] it is proved that there are not algebraic limit cycles except for  $\ell N \delta \neq 0$  and  $M^2 - 4\ell N \geq 0$  in family (I) (this result is improved in [15] where it is proved that there is no algebraic limit cycle for family (I)). Moreover, they also prove that the polynomial inverse integrating factors into families (I),  $(II)_{N=0}$ ,  $(III)_{a=0}$  and  $(III)_{N=0}$  generically have at most degree 3. So, in the studied cases, the existence of polynomial inverse integrating factor implies the nonexistence of limit cycles or at most the existence of a circle as a unique limit cycle.

Another interesting example of application of Theorem 25 is given in the proof of several extensions to the Bendixson-Dulac Criterion to study of the number of limit cycles of planar differential systems, see [57, 58, 59]. An open set  $\mathcal{U} \subseteq \mathbb{R}^2$  with smooth boundary is said to be  $\ell$ -connected if its fundamental group,  $\pi_1(\mathcal{U})$  is  $Z * \dots^{(\ell)} * Z$ , or in other words if  $\mathcal{U}$  has  $\ell$  gaps. The classical Bendixson-Dulac Criterion is the following proposition, see [57] for the statement and a short proof.

**Proposition 26** (Bendixson-Dulac Criterion) *Let  $\mathcal{U}$  be an open  $\ell$ -connected subset of  $\mathbb{R}^2$  with smooth boundary. Let  $\mathcal{X} = P(x, y) \partial_x + Q(x, y) \partial_y$  be a vector field*

of class  $\mathcal{C}^1$  in  $\mathcal{U}$ . Let  $g : \mathcal{U} \rightarrow \mathbb{R}$  be a  $\mathcal{C}^1$  function such that

$$M := \operatorname{div}(g\mathcal{X}) = P \frac{\partial g}{\partial x} + Q \frac{\partial g}{\partial y} + g \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right)$$

does not change sign in  $\mathcal{U}$  and vanishes only on a null measure Lebesgue set, such that  $\{M = 0\} \cap \{g = 0\}$  does not contain periodic orbits of  $\mathcal{X}$ . Then the maximum number of periodic orbits of  $\mathcal{X}$  contained in  $\mathcal{U}$  is  $\ell$ . Furthermore, each one of them is a hyperbolic limit cycle that does not cut  $\{g = 0\}$  and its stability is given by the sign of  $gM$  over it.

## 7 The zero set of inverse integrating factors

In Theorem 25 it is shown that limit cycles  $\gamma$  of a  $\mathcal{C}^1$  real planar vector field  $\mathcal{X}$  belong to the zero set of any inverse integrating factor of  $\mathcal{X}$  defined near  $\gamma$ , that is,  $\gamma \subset V^{-1}(0)$ .

In addition to containing any limit cycle of  $\mathcal{X}$  lying in  $\mathcal{U}$ , the zero set of  $V$  is also often connected to the separatrices of critical points of  $\mathcal{X}$  in  $\mathcal{U}$ . To understand why, recall that integral curves of  $\mathcal{X}$  that map to themselves under the action of a Lie group are *invariant solutions* for the Lie group. Recall that when  $\mathcal{Y} = \xi(x, y)\partial_x + \eta(x, y)\partial_y$  is the infinitesimal generator of a nontrivial local Lie group of symmetries of  $\mathcal{X}$  then the function  $V(x, y) = \det\{\mathcal{X}, \mathcal{Y}\}$  is an inverse integrating factor of  $\mathcal{X}$ , as it has already been stated in Section 3. It is obvious that every solution of  $\mathcal{X}$  which remains invariant under the action of the group with infinitesimal generator  $\mathcal{Y}$  must satisfy  $V(x, y) = 0$ . In other words, inverse integrating factors must vanish on invariant solutions.

Based on these ideas, Bluman and Anco [6] argue heuristically that separatrices should also lie in  $V^{-1}(0)$ . Of course, any saddle loop in a Hamiltonian system is composed of separatrices not lying in the zero set of the trivial inverse integrating factor  $V \equiv 1$ . Nevertheless, the idea has merit and we expect the zero set of  $V$  to play a role in the dynamics of  $\mathcal{X}$  and it is very surprising that this fact was not completely accomplished until recent times. We repeat verbatim the following historical development on this issue given in [3].

“To our best knowledge, J. M. Page was the first author in making an observation of this kind. Concretely, the idea is developed in [88] of using Lie groups in the computation of singular solutions to the implicit first order differential equation  $F(x, y, y') = 0$ . The same idea is gathered in pgs. 113 and ss. of [88], where several examples of calculation of envelopes are given, and later quoted without variations in [35], pgs. 66 and ss.. In pg. 111 of his classical textbook [75], first published in

1926, E. L. Ince rescue Page's observation on envelopes but no other material is added. It would took several decades until some advance along this line of thinking might be registered. In this regard, the works of W. H. Steeb, C. E. Wulfman and G. D. Bluman and S. Kumei must be cited. In [102], Steeb discussed the connection between limit cycles of two-dimensional systems and one-parameter groups of transformations. In [107], Wulfman stated apparently general conditions on the infinitesimal generator of a Lie group admitted by a system of autonomous differential equations in order that an invariant solution is a limit cycle of the system. However, the argument he offers to support these conditions rests on a heuristic more than rigorous basis. In turn, chapter 3 of [7] contains a section (Section 3.6) devoted to discuss the relationships existing between invariant solutions on one hand and exceptional paths on the other. Even though the developments in this section of the book seems to remain also on a semi-heuristic level, several examples and exercises are provided showing how the technique works in particular systems."

In addition to this exhaustive historical description, we also would like to add the work of González–Gascón [71] where it is pointed out that if there is an infinitesimal generator of a Lie symmetry  $\mathcal{Y}$  of a vector field  $\mathcal{X}$  in  $\mathbb{R}^n$ , then on the limit cycles (periodic isolated orbits) of  $\mathcal{X}$  it follows that  $\mathcal{X}$  and  $\mathcal{Y}$  are parallel. This implies, in the particular case of planar fields that the associated inverse integrating factor  $V = \det\{\mathcal{X}, \mathcal{Y}\}$  vanishes on the limit cycle.

In [3] Berrone and Giacomini showed that, under mild additional hypotheses, the separatrices of *hyperbolic* saddle–points lying in  $\mathcal{U}$  are contained in  $V^{-1}(0)$ , and extended this result by showing that if  $\Gamma$  is a compact limit set all of whose critical points are hyperbolic saddle–points, then under mild conditions  $\Gamma \subset V^{-1}(0)$  holds. Now, we summarize the results in [3].

It is easy to see that isolated vanishing points of an inverse integrating factor are singular points of the vector field. Moreover, for non–degenerate singularities (singularities  $p_0 \in \mathbb{R}^2$  of  $\mathcal{X}$  with non vanishing Jacobian determinant  $\det(D\mathcal{X}(p_0)) \neq 0$ ) one has the following result.

**Theorem 27** [3] *Let  $p_0$  be a non–degenerate critical point of a  $C^1$  vector field  $\mathcal{X}$  and let  $V$  be an inverse integrating factor defined in a neighborhood of  $p_0$  and satisfying  $V(p_0) \neq 0$ . If  $\det(D\mathcal{X}(p_0)) > 0$  then  $p_0$  is a center. On the contrary, when  $\det(D\mathcal{X}(p_0)) < 0$ ,  $p_0$  is a saddle–point.*

Next theorem is concerned with the stability of isolated zeroes of an inverse integrating factor.

**Theorem 28** [3] *Let  $p_0$  be an isolated zero of a non-negative inverse integrating factor  $V$  of a  $\mathcal{C}^1$  vector field  $\mathcal{X}$  defined in a neighborhood  $\mathcal{U}$  of  $p_0$ . Then  $p_0$  is a stable (resp. unstable) singular point of  $\mathcal{X}$  provided that  $\operatorname{div}\mathcal{X}|_{\mathcal{U}} \leq 0$  (resp.  $\geq 0$ ). Furthermore,  $p_0$  is asymptotically stable (resp. unstable) provided that  $\operatorname{div}\mathcal{X}|_{\mathcal{U}} < 0$  (resp.  $> 0$ ).*

When a singularity  $p_0$  of  $\mathcal{X}$  is a non-isolated zero of an inverse integrating factor the following result holds. Here, given an orbit  $\gamma_0$  of  $\mathcal{X}$ , we denote by  $\omega(\gamma_0)$  and  $\alpha(\gamma_0)$  its  $\omega$ -limit set and  $\alpha$ -limit set respectively.

**Theorem 29** [3] *Let  $p_0$  be a non-isolated zero of an inverse integrating factor  $V$  of a  $\mathcal{C}^1$  vector field  $\mathcal{X}$ . Then, one of the following two possibilities may occur:*

- (a) *There exists at least an orbit  $\gamma_0$  of  $\mathcal{X}$  (different of  $p_0$ ) such that  $\omega(\gamma_0) = p_0$  or  $\alpha(\gamma_0) = p_0$  and  $V|_{\gamma_0} \equiv 0$ .*
- (b) *There exists a infinite sequence  $\{\gamma_n\}_{n \in \mathbb{N}}$  of periodic orbits of  $\mathcal{X}$  accumulating at  $p_0$  such that  $V|_{\gamma_n} \equiv 0$ .*

A singularity  $p_0$  of the vector field  $\mathcal{X}$  is called *strong* if  $\operatorname{div}\mathcal{X}(p_0) \neq 0$ . Otherwise, when  $\operatorname{div}\mathcal{X}(p_0) = 0$ , it is called *weak*. For a linear strong saddle points, it is easy to see that every inverse integrating factor must vanish on all four separatrix curves of the saddle. As it is established by the next theorem, the situation with nonlinear hyperbolic saddle points of  $\mathcal{C}^1$  systems is entirely analogous to the linear case. The proof is based on the normal form of  $\mathcal{X}$  near the hyperbolic saddle  $p_0$  and the Stable Manifold Theorem.

**Theorem 30** [3] *Let  $p_0$  be a hyperbolic saddle-point of a  $\mathcal{C}^1$  vector field  $\mathcal{X}$  and  $V$  an inverse integrating factor defined in a neighborhood  $\mathcal{U}$  of  $p_0$ . Then  $V$  vanishes on all four separatrix curves of the saddle provided that one of the following conditions holds: (i)  $p_0$  is strong; (ii)  $p_0$  is weak and  $V(p_0) = 0$ .*

In the work [50], the previous theorem is slightly improved. If  $p_0$  is a hyperbolic saddle point of a  $\mathcal{C}^{k+1}$  vector field  $\mathcal{X}$  whose  $k^{th}$  saddle quantity is not zero and  $V$  is an inverse integrating factor defined in a neighborhood of  $p_0$ , then  $V(p_0) = 0$  (and, thus,  $V$  vanishes on all four separatrix curves of the saddle). For a full definition of saddle quantities see Subsection 8.2 in relation with system (23).

As a corollary of Theorem 30, one can ensure the vanishing of an inverse integrating factor defined near certain saddle connections. Recall that a saddle connection is a union of saddle points and orbits connecting them.

**Corollary 31** [3] *Let  $V$  be an inverse integrating factor defined in a region containing a saddle connection  $\Gamma$  whose critical points are  $p_i$  for  $i = 1, \dots, n$ . If  $V$  vanishes at a certain singular point  $p_k$ , then  $V|_{\Gamma} \equiv 0$ .*

A graphic  $\bar{\Gamma} = \cup_{i=1}^k \phi_i(t) \cup \{p_1, \dots, p_k\}$  is formed by  $k$  singular points  $p_1, \dots, p_k$ ,  $p_{k+1} = p_1$  and  $k$  oriented regular orbits  $\phi_1(t), \dots, \phi_k(t)$ , connecting them such that  $\phi_i(t)$  is an unstable characteristic orbit of  $p_i$  and a stable characteristic orbit of  $p_{i+1}$ . A graphic may or may not have associated a Poincaré return map. In case it has one, it is called a *polycycle*.

Now, let us suppose that  $\Gamma$  is a *graphic*, that is,  $\Gamma$  is a limit set which differs from a critical point or a periodic orbit.

**Theorem 32** [3] *Let  $V$  be an inverse integrating factor defined in a region containing a compact graphic  $\Gamma$ . Then, the following holds:*

- (a)  *$V$  vanishes at a critical point at least of  $\Gamma$ .*
- (b) *If all the critical points on  $\Gamma$  are non-degenerate, then  $V|_{\Gamma} \equiv 0$ .*

The main results of the paper [56] are generalizations and extensions of the previous results stated in [3]. A key ingredient in the proof of the results of [56] is the concept of an integral invariant, introduced by Poincaré in [93] for arbitrary dimension, and its relation to inverse integrating factors. We denote by  $\phi(t; (x_0, y_0))$  the solution of (3) passing through the point  $(x_0, y_0) \in \mathcal{U}$  at  $t = 0$ ;  $\phi(t; D)$  will denote the image of a domain  $D \subset \mathcal{U}$  under the time- $t$  map of the flow generated by the solutions of system (3).

**Definition 33** *Let  $\mu : \mathcal{U} \subset \mathbb{R}^2 \rightarrow \mathbb{R}$  be a non-zero integrable function on  $\mathcal{U}$ . The integral*

$$\int_{\phi(t; D)} \mu(x, y) \, dx dy \tag{16}$$

*is an integral invariant of system (3) if for any measurable set  $D \subset \mathcal{U}$  the integral is independent of  $t$ .*

The function  $\mu$  is called the *density* of the integral invariant, based on the obvious hydrodynamic interpretation. Various versions of the following result can be found in textbooks, see for instance [2]. We state it in a form suited to our needs. In [56] is also provided a short proof.

**Lemma 34** [93] *Let  $\mathcal{U}$  be an open subset of  $\mathbb{R}^2$ , let  $V : \mathcal{U} \rightarrow \mathbb{R}$  be a  $\mathcal{C}^1$  function, and define a  $\mathcal{C}^1$  function  $\mu : \mathcal{U} \setminus V^{-1}(0) \rightarrow \mathbb{R}$  by  $\mu = 1/V$ . Then  $V$  is an inverse integrating factor of system (3) in  $\mathcal{U}$  if and only if the integral (16) is an integral invariant for system (3) on  $\mathcal{U} \setminus V^{-1}(0)$ .*

By using the relationship between inverse integrating factors and integral invariants given in Lemma 34, it is easy to see the next result. The definition of parabolic or elliptic sector can be found, for instance, in [44].

**Theorem 35** [56] *Let  $p_0$  be any critical point of system (3) at which there is an elliptic or parabolic sector. If  $V$  is any inverse integrating factor of (3) defined on a neighborhood of  $p_0$ , then  $V(p_0) = 0$ .*

In order to state the next result, we recall that a function  $f$  is called a *Morse function* if all its critical points are nondegenerate, i.e., the associated Hessian matrix has maximal rank at all the critical points. For Morse functions it is well known, see [74] for instance, that the set of critical points is discrete, that is, has no accumulation points.

**Theorem 36** [56] *Let  $\Gamma$  be any compact  $\alpha$ - or  $\omega$ -limit set of system (3) that contains a regular point, and let  $V$  be any inverse integrating factor of (3) defined in some neighborhood of  $\Gamma$ . Depending on the smoothness of  $V$ , the following statements hold.*

- (a) *There exists a point  $p$  in  $\Gamma$  such that  $V(p) = 0$ .*
- (b) *If  $V$  is  $\mathcal{C}^2$ , then either  $\Gamma$  contains a point that is an accumulation point of isolated critical points of  $V$  or  $\Gamma \subset V^{-1}(0)$ .*
- (c) *If  $V$  is real analytic or Morse, then  $\Gamma \subset V^{-1}(0)$ .*

Theorem 30 does not hold, in general, for non-hyperbolic singularities. But it does generalize for saddle or saddle-node singularities with exactly one non-zero eigenvalue, as the next result shows.

**Theorem 37** [56] *Suppose  $p_0$  is an isolated singularity of a  $\mathcal{C}^1$  vector field  $\mathcal{X}$ , and that  $V$  is an inverse integrating factor for  $\mathcal{X}$  defined in a neighborhood of  $p_0$ . If the linear part  $D\mathcal{X}(p_0)$  has exactly one zero eigenvalue, then  $V$  vanishes along any separatrix of  $\mathcal{X}$  at  $p_0$ .*

We finish this section by stating a corollary of Theorems 30 and 37.

**Corollary 38** [56] *Let  $\Gamma$  be a polycycle (or graphic which need not be a limit set) of system (3) and let  $V$  be any inverse integrating factor of (3) defined in some neighborhood of  $\Gamma$ . Assume that the critical points of (3) that belong to  $\Gamma$  are hyperbolic saddles  $p_1, p_2, \dots, p_n$  or saddles and saddle-nodes  $q_1, q_2, \dots, q_m$  with exactly one zero eigenvalue. If the separatrices of  $\Gamma$  are such that they always connect either  $p_k$  with  $p_j$  and  $V(p_k) = 0$  or  $p_k$  with  $q_j$  or  $q_k$  with  $q_j$  then  $\Gamma \subset V^{-1}(0)$ .*

As the authors of [56] remark, the hypothesis in Theorem 36 that  $V$  be real analytic does not seem to be essential. Thus, in [56] it is conjectured that only in the class  $\mathcal{C}^1$  for  $V$ , Theorem 36 remains valid. This conjecture was solved positively in [47]. In summary, in that paper the authors prove that there always exists a smooth inverse integrating factor in a neighborhood of a limit cycle and obtain a necessary and sufficient condition for the existence of an analytic one. This condition is expressed in terms of the Ecalle–Voronin modulus of the associated Poincaré map. We recall that a germ of a map in the set of real analytic diffeomorphisms near the origin of  $\mathbb{R}$  is analytically embeddable, i.e., it is the time-one map of an analytic vector field on the line, if and only if its Ecalle–Voronin modulus is trivial. The embedding properties of the Poincaré map are crucial for the proof of the next theorem.

**Theorem 39** [47] *Let  $\gamma$  be a limit cycle of the analytic planar vector field  $\mathcal{X}$ . Then there exists a neighborhood  $\mathcal{U}$  of  $\gamma$  and a function  $V \in \mathcal{C}^\infty(\mathcal{U})$  which is an inverse integrating factor of  $\mathcal{X}$  and vanishes exactly on  $\gamma$ . Moreover,  $V$  can be chosen analytic if and only if the Ecalle–Voronin modulus of the germ of the Poincaré map of  $\mathcal{X}$  along the limit cycle  $\gamma$  is trivial.*

**Corollary 40** [47] *If  $\gamma$  is a hyperbolic limit cycle of an analytic vector field  $\mathcal{X}$ , then  $\mathcal{X}$  admits an analytic inverse integrating factor in a neighborhood of  $\gamma$ .*

In addition, in [47] it is also proved that a  $\mathcal{C}^1$  inverse integrating factor of a  $\mathcal{C}^1$  planar vector field must vanish identically on the polycycles which are limit sets of its flow. We recall that a polycycle of a  $\mathcal{C}^1$  vector field is a compact invariant set which contains both regular and singular points.

**Theorem 41** [47] *Let  $\mathcal{X}$  be a  $\mathcal{C}^1$  vector field defined in a domain  $\mathcal{U} \subseteq \mathbb{R}^2$ . Suppose that  $\Gamma \subset \mathcal{U}$  is a polycycle which is a limit set of  $\mathcal{X}$  and  $\mathcal{X}$  has a finite number of singular points in  $\Gamma$ . Then if  $\mathcal{X}$  admits a  $\mathcal{C}^1$  inverse integrating factor  $V$  in  $\mathcal{U}$ , then  $\Gamma \subset V^{-1}(0)$ .*

The main idea of the proof of Theorem 41 is to pull back the vector field  $\mathcal{X}$  and the inverse integrating factor  $V$  to the universal cover of  $\mathcal{U} \setminus \{V\mathcal{X} = 0\}$  and exploit the fact that  $\mathcal{X}/V$  lifts to a Hamiltonian vector field in the covering space.

The existence of inverse integrating factors in a neighborhood of an elementary singularity is also established in [47]. The regularity of the inverse integrating factor depends on the kind of singularity and the proof makes crucial use of the theory of normal forms for planar vector fields. This considerably extends previous results in [21], where the authors prove for analytic vector fields the existence of a unique analytic inverse integrating factor in a neighborhood of a strong focus, or a non-resonant hyperbolic node, or a Siegel hyperbolic saddle.



The following result, which is stated in [51], is a summary and a generalization of several results on the existence of a smooth and non-flat inverse integrating factor  $V_0(x, y)$  in a neighborhood of an isolated singular point, see [21, 47, 67].

**Theorem 42** *Let the origin be an isolated singular point of (3) and let  $\lambda, \mu \in \mathbb{C}$  be the eigenvalues associated to the linear part of (3). If  $\lambda \neq 0$ , then there exists a smooth and non-flat inverse integrating factor  $V(x, y)$  in a neighborhood of the origin.*

In [47] the existence of an analytic inverse integrating factor in a neighborhood of a non-degenerate monodromic singular point of an analytic system is characterized. If the origin is a non-degenerate center or a strong focus, there exists an analytic inverse integrating factor. If the origin is a weak focus, by Theorem 42 we have the existence of a smooth and non-flat inverse integrating factor, and there exists an analytic inverse integrating factor if and only if the Ecalle–Voronin modulus of the associated Poincaré map is trivial, see also Theorem 39. In [47] the first known examples of real planar analytic vector fields not admitting an analytic inverse integrating factor in any neighborhood of either a limit cycle or an isolated singularity are given.

In [51], we show the existence of an inverse integrating factor in a neighborhood of some degenerate singular points.

**Theorem 43** [51] *There exists an inverse integrating factor  $V(x, y)$ , of class at least  $\mathcal{C}^1$ , in a neighborhood of the following two types of singular points: a degenerate focus without characteristic directions and a nilpotent focus.*

## 8 Bifurcations

The inverse integrating factor has been shown to be very useful in many bifurcation problems. The books [73, 97] contain the main concepts and ideas of this theory in the framework of ordinary differential equations.

Consider system (3) and take a parametric family of systems of the form

$$\dot{x} = \mathcal{P}(x, y, \varepsilon), \quad \dot{y} = \mathcal{Q}(x, y, \varepsilon), \quad (17)$$

where  $\mathcal{P}(x, y, \varepsilon)$  and  $\mathcal{Q}(x, y, \varepsilon)$  are analytic functions in  $(x, y)$  in the same open set as  $P(x, y)$  and  $Q(x, y)$  (or an open set we are interested in), are analytic for  $\varepsilon$  near the origin and coincide with  $P(x, y)$  and  $Q(x, y)$  when  $\varepsilon = 0$ , that is,  $\mathcal{P}(x, y, 0) = P(x, y)$  and  $\mathcal{Q}(x, y, 0) = Q(x, y)$ . The parameter  $\varepsilon$  is called *bifurcation parameter* and we assume it is defined in a neighborhood of the origin of  $\mathbb{R}^k$ , with  $k \in \mathbb{N}$ ; in many cases we consider that  $\varepsilon$  is a real one-dimensional parameter ( $k = 1$ ). For small values of the norm of  $\varepsilon$ , we say that the family of systems (17) is a

perturbation of system (3). When  $\varepsilon$  takes values near the origin  $0 < |\varepsilon| \ll 1$ , the qualitative behavior of system (17) can change with respect to the one of system (3) for  $\varepsilon = 0$ . In this case, we say that a bifurcation has occurred. Bifurcation theory aims at characterizing under which conditions on system (3) and its perturbations, this bifurcations eventually happen and which are their properties. For example, consider a singular point  $p$  of system (3) and denote by  $\lambda$  and  $\mu$  the eigenvalues of the linearization of the system around  $p$ . If  $\lambda, \mu \in \mathbb{R}$  and  $\lambda \cdot \mu < 0$ , then we say that  $p$  is a hyperbolic saddle and a classical result states that any perturbation of the system in a neighborhood of this point has the same qualitative behavior, that is, we have a saddle singular point that, when  $\varepsilon$  tends to zero, tends to  $p$ . We give the adjective *hyperbolic* to those objects which maintain their qualitative nature under perturbations. In contrast, if  $p$  is a singular point of center type, i.e. it has a neighborhood filled with periodic orbits, then a perturbation of the system usually breaks these orbits and the point can be transformed, for instance, into a singular point of focus type, i.e. surrounded by orbits that spiral towards (or from) it. In this case, we say that  $p$  is a bifurcation point. When the considered family (17) shows all the possible sample of qualitative behaviors that might occur when perturbing an object of system (3), we say that it is an *unfolding*. The minimum number of parameters needed to have an unfolding is called the *codimension*.

Bifurcation theory is one of the most current tools used when trying to solve 16<sup>th</sup> Hilbert problem, part b. This problem was proposed in 1900 by D. Hilbert and asks for the maximum number and possible configurations of limit cycles that a polynomial system of the form (7) of degree  $d$  may have, only depending on the degree  $d$ . For a fixed system, Écalle (1992) and Il'yashenko (1991) have demonstrated, in a different and independent way, that the number of limit cycles that the system may have is finite. However, the problem of determining whether there exists an upper bound on the number of limit cycles that a polynomial system of the form (7) can have, only depending on the degree  $d$  of the system, is still open.

As R. Roussarie defines in [97], given a family of systems of the form (17), a *limit periodic set* is a compact and nonempty subset  $\Gamma$  of points so that there exists some succession  $(\varepsilon_n)_n$  which tends to  $\varepsilon_*$  when  $n \rightarrow +\infty$  such that for every  $\varepsilon_n$ , the corresponding system (17) has a limit cycle  $\gamma_{\varepsilon_n}$  which tends to  $\Gamma$ , in the sense of the Hausdorff distance, when  $n \rightarrow +\infty$ . In this context, it is assumed that the parameters take values in a compact set. Following an analogous argument to the one used to prove Poincaré–Bendixson Theorem, the structure of limit periodic sets can be determined. Given a limit periodic set of the family (17), we define its *cyclicity* as the maximum number of limit cycles which can be bifurcated from  $\Gamma$  in this family. In [96], see also [97], R. Roussarie showed that the existence of a uniform upper bound in the number of limit cycles of an analytic family (17) is equivalent to that each of its limit periodic sets  $\Gamma$  has finite cyclicity. This equivalence and the fact that all the limit periodic sets in a family (17) can be

determined shows how bifurcation theory allows to tackle 16<sup>th</sup> Hilbert problem. In [46], Dumortier, Roussarie and Rousseau established a list of 121 cases which are all the possible limit periodic sets that can appear within the family of quadratic systems and proposed a program, currently unfinished, to study all these graphics to demonstrate that there is a uniform upper bound for the number of limit cycles of polynomial systems of degree 2.

The knowledge of inverse integrating factors for particular systems has simplified its study and has allowed the understanding of several bifurcations. In [19], for instance, the following family of cubic systems

$$\begin{aligned}\dot{x} &= \lambda x - y + \lambda m_1 x^3 + (m_2 - m_1 + m_1 m_2) x^2 y + \lambda m_1 m_2 x y^2 + m_2 y^3, \\ \dot{y} &= x + \lambda y - x^3 + \lambda m_1 x^2 y + (m_1 m_2 - m_1 - 1) x y^2 + \lambda m_1 m_2 y^3,\end{aligned}$$

where  $\lambda$ ,  $m_1$  and  $m_2$  are arbitrary real parameters, is considered. The fact of knowing an inverse integrating factor

$$V(x, y) := (x^2 + y^2)(1 + m_1 x^2 + m_1 m_2 y^2)$$

for this family of systems allows the determination of all the bifurcations within the family.

Indeed, inverse integrating factors allow the understanding of the bifurcation of limit cycles from many limit periodic sets, as we explain in this section. The main result used in this context is Theorem 25 as it states that any inverse integrating factor defined in a neighborhood of a limit cycle needs to vanish on it. We recall that the zero set of a limit cycle is formed by orbits of the system.

We split this section in three subsections depending on the considered limit periodic sets.

## 8.1 Bifurcation from a period annulus

In this subsection we consider planar differential systems of the form (3) with a singular point of center type. The set of periodic orbits surrounding this point is called its *period annulus*. A perturbation of the system usually breaks these periodic orbits but some of them might be maintained as limit cycles for the perturbed system. We say that this periodic orbits have bifurcated from the period annulus.

There are several methods to determine how many limit cycles bifurcate from the periodic orbits of a period annulus. These methods are based upon different tools: the Poincaré return map, see for instance [5]; the Poincaré-Pontrjagin-Melnikov integrals, see for instance [72]; averaging theory, see [98]; and the inverse integrating factor, see [64, 65, 103]. This last method also gives the shape of the

bifurcated limit cycles up to any order of the perturbation parameter.

We are going to explain the method described in [64]. Let us consider a Hamiltonian planar differential system with a center at the origin:

$$\dot{x} = \frac{\partial H}{\partial y}, \quad \dot{y} = -\frac{\partial H}{\partial x},$$

where  $H(x, y)$  is the Hamiltonian function and it is analytic in a neighborhood of the origin. We denote by  $\mathcal{P}$  the period annulus of the center at the origin. Any analytic (nonzero) function of the Hamiltonian is an inverse integrating factor of the system. In particular, any (nonzero) constant function is an inverse integrating factor.

Let us consider an analytic perturbation of the previous Hamiltonian system:

$$\dot{x} = \mathcal{P}(x, y, \varepsilon), \quad \dot{y} = \mathcal{Q}(x, y, \varepsilon), \quad (18)$$

where

$$\mathcal{P}(x, y, \varepsilon) := \frac{\partial H}{\partial y} + \sum_{k=1}^{\infty} \varepsilon^k f_k(x, y), \quad \mathcal{Q}(x, y, \varepsilon) := -\frac{\partial H}{\partial x} + \sum_{k=1}^{\infty} \varepsilon^k g_k(x, y),$$

and where  $\varepsilon$  is a small real parameter and  $f_k(x, y)$ ,  $g_k(x, y)$  are analytic functions in  $\mathcal{P} \cup \{(0, 0)\}$ . Let us look for an analytic solution

$$V(x, y, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k V_k(x, y),$$

of the partial differential equation

$$\mathcal{P} \frac{\partial V}{\partial x} + \mathcal{Q} \frac{\partial V}{\partial y} = \left( \frac{\partial \mathcal{P}}{\partial x} + \frac{\partial \mathcal{Q}}{\partial y} \right) V.$$

This partial differential equation gives a succession of linear differential equations for the functions  $V_k(x, y)$  which can be solved recursively.

The equation of order 0 in  $\varepsilon$  implies that  $V_0(x, y)$  needs to be an inverse integrating factor for the unperturbed system. We have, thus, that  $V_0 = V_0(H)$  is a function of the Hamiltonian  $H(x, y)$ .

The equation of order 1 in  $\varepsilon$  gives a linear differential equation in  $V_1(x, y)$  whose non-homogeneous term contains the function  $V_0(x, y)$ . Imposing that the function  $V$  needs to be periodic when evaluated on the unperturbed periodic orbits, it can be shown that  $V_0(h)$  needs to be

$$V_0(h) = \lambda \int_{\{H=h\}} g_1(x, y) dx - f_1(x, y) dy,$$

where  $\lambda$  is a nonzero real constant, and  $\{H = h\}$  denotes the periodic orbit of  $\mathcal{P}$  contained in the  $h$ -level set of the Hamiltonian  $H(x, y)$ . Therefore,  $V_0(h)$  is the first Poincaré-Pontrjagin-Melnikov integral associated to system (18). Once we take this expression of  $V_0$ , we can solve the linear differential equation for  $V_1$  which is determined up to the sum of an arbitrary function of the Hamiltonian  $H$ .

By induction on  $k$  it can be shown that  $V_k$  is determined up to the sum of an arbitrary function of the Hamiltonian  $H$  which we denote by  $W_k(h)$ . When solving the linear differential equation of order  $k + 1$  in  $\varepsilon$ , and imposing that the function  $V$  needs to be periodic on the orbits of  $\mathcal{P}$ , it can be shown that  $W_k(h)$  corresponds to the  $k + 1$  Poincaré-Pontrjagin-Melnikov integral associated to system (18).

Indeed, in [64], the authors show that, fixed a small value of  $|\varepsilon|$ , the zero sets of the functions  $\sum_{k=0}^n \varepsilon^k V_k(x, y)$  give approximations up to order  $\varepsilon^n$  of the limit cycles of system (18) which bifurcate from  $\mathcal{P}$ . When increasing the value of  $n$ , better approximations of these limit cycles are obtained and, thus, their shape is determined.

In [65], this method is generalized to non-Hamiltonian centers. The paper [103] purports a better understanding of this method as it studies this problem when the first  $\ell - 1$  Poincaré-Pontrjagin-Melnikov functions are identically zero. The main result in this paper is that, in this case,  $V_0(h)$  is the first non identically zero Poincaré-Pontrjagin-Melnikov function.

Most of these ideas are also used in [68] to determine semistable limit cycles that bifurcate from  $\mathcal{P}$ . Moreover, the method is applied to study the limit cycles which bifurcate from a Liénard system.

We remark that this method is not only an alternative to the other methods as it shows how the inverse integrating factor is linked to bifurcation problems. This method is computationally as difficult as any other method but, moreover, it provides the shape of the bifurcated limit cycles.

## 8.2 Bifurcation from monodromic $\omega$ -limit sets

The work [50] is concerned with planar real analytic systems (3) with an analytic inverse integrating factor defined in a neighborhood of a regular orbit  $\phi(t)$ . First of all it is shown that the inverse integrating factor defines an ordinary differential equation for the transition map along the orbit, see equation (21). Taking two transversal sections  $\Sigma_1$  and  $\Sigma_2$  based on  $\phi(t)$ , it is studied the transition map of the flow of  $\mathcal{X}$  in a neighborhood of  $\phi(t)$ . This transition map is studied by means of the *Poincaré map*  $\Pi : \Sigma_1 \rightarrow \Sigma_2$ . Given a point in  $\Sigma_1$ , we consider the orbit of (3) with it as initial point and we follow this orbit until it first intersects  $\Sigma_2$ .

Let  $(\varphi(s), \psi(s)) \in \mathcal{U}$ , with  $s \in \mathcal{I} \subseteq \mathbb{R}$  be a parameterization of the regular orbit

$\phi(t)$  between the base points of  $\Sigma_1$  and  $\Sigma_2$ . Given a point  $(x, y)$  in a sufficiently small neighborhood of the orbit  $(\varphi(s), \psi(s))$ , we can always encounter values of the *curvilinear coordinates*  $(s, n)$  that realize the following change of variables:  $x(s, n) = \varphi(s) - n\psi'(s)$ ,  $y(s, n) = \psi(s) + n\varphi'(s)$ . We remark that the variable  $n$  measures the distance perpendicular to  $\phi(t)$  from the point  $(x, y)$  and, therefore,  $n = 0$  corresponds to the considered regular orbit  $\phi(t)$ . We can assume, without loss of generality, that the transversal section  $\Sigma_1$  corresponds to  $\Sigma_1 := \{s = 0\}$  and  $\Sigma_2$  to  $\Sigma_2 := \{s = L\}$ , for a certain real number  $L > 0$ . We perform the change to curvilinear coordinates  $(x, y) \mapsto (s, n)$  in a neighborhood of the regular orbit  $n = 0$  with  $s \in \mathcal{I} = [0, L]$ . Then, system (3) is written as the following ordinary differential equation:

$$\frac{dn}{ds} = F(s, n). \quad (19)$$

We denote by  $\Psi(s; n_0)$  the flow associated to the equation (19) with initial condition  $\Psi(0; n_0) = n_0$ . In these coordinates, the Poincaré map  $\Pi : \Sigma_1 \rightarrow \Sigma_2$  between these two transversal sections is given by  $\Pi(n_0) = \Psi(L; n_0)$ .

We assume the existence of an analytic inverse integrating factor  $V(x, y)$  in a neighborhood of the considered regular orbit  $\phi(t)$  of the analytic system (3). In fact, when  $\Sigma_1 \neq \Sigma_2$  and no return is involved, there always exists such an inverse integrating factor. The change to curvilinear coordinates gives us an inverse integrating factor for equation (19), denoted by  $\tilde{V}(s, n)$  and which satisfies

$$\frac{\partial \tilde{V}}{\partial s} + \frac{\partial \tilde{V}}{\partial n} F(s, n) = \frac{\partial F}{\partial n} \tilde{V}(s, n). \quad (20)$$

Now, we can state one of the main results of [50].

**Theorem 44** [50] *We consider a regular orbit  $\phi(t)$  of the analytic system (3) which has an inverse integrating factor  $V(x, y)$  of class  $\mathcal{C}^1$  defined in a neighborhood of it and we consider the Poincaré map associated to the regular orbit between two transversal sections  $\Pi : \Sigma_1 \rightarrow \Sigma_2$ . We perform the change to curvilinear coordinates and we consider the ordinary differential equation (19) with the inverse integrating factor  $\tilde{V}(s, n)$  which is obtained from  $V(x, y)$ . In these coordinates, the transversal sections can be taken such that  $\Sigma_1 := \{s = 0\}$  and  $\Sigma_2 := \{s = L\}$ , for a certain real value  $L > 0$ . We parameterize  $\Sigma_1$  by the real value of the coordinate  $n$ . The following identity holds.*

$$\tilde{V}(L, \Pi(n)) = \tilde{V}(0, n) \Pi'(n). \quad (21)$$

Theorem 44 is the key point to prove Theorems 46 and 48.

Further, in [50] the authors consider regular orbits whose Poincaré map is a return map and take profit from the result stated in Theorem 44 in order to study

the Poincaré map associated to a limit cycle or to a homoclinic loop, in terms of the inverse integrating factor. To do that, the following definition of *vanishing multiplicity* of an analytic inverse integrating factor  $V(x, y)$  of the analytic system (3) over a regular orbit  $\phi(t)$  is needed.

**Definition 45** [50] *Let  $V(x, y)$  be an analytic inverse integrating factor of the analytic system (3) and  $\phi(t)$  a regular orbit of it parameterized by  $(\varphi(s), \psi(s)) \in \mathcal{U}$ , with  $s \in \mathcal{I} \subseteq \mathbb{R}$ . Consider the local change of coordinates  $x(s, n) = \varphi(s) - n\psi'(s)$ ,  $y(s, n) = \psi(s) + n\varphi'(s)$  defined in a neighborhood of the considered regular orbit  $n = 0$  and take the following Taylor development around  $n = 0$ :*

$$V(x(s, n), y(s, n)) = n^m v(s) + O(n^{m+1}), \quad (22)$$

where  $m$  is an integer with  $m \geq 0$  and the function  $v(s)$  is not identically null, we say that  $V$  has multiplicity  $m$  on  $\phi(t)$ .

In fact, in [50] it is proved that  $v(s) \neq 0$  for any  $s \in \mathcal{I}$ , and thus, the vanishing multiplicity of  $V$  on  $\phi(t)$  is well-defined over all its points.

Let us consider as regular orbit a limit cycle  $\gamma$  and we use the parameterization of  $\gamma$  in curvilinear coordinates  $(s, n)$  with  $s \in [0, L)$ . Thus, the Poincaré map associated to  $\gamma$  is  $\Pi(n_0) = \Psi(L; n_0)$ . It is well known that  $\Pi$  is analytic in a neighborhood of  $n_0 = 0$ . We recall that the periodic orbit  $\gamma$  is a limit cycle if, and only if, the Poincaré return map  $\Pi$  is not the identity. If  $\Pi$  is the identity, we have that  $\gamma$  belongs to a period annulus. We recall the definition of multiplicity of a limit cycle:  $\gamma$  is said to be a limit cycle of *multiplicity 1* if  $\Pi'(0) \neq 1$  and  $\gamma$  is said to be a limit cycle of multiplicity  $m$  with  $m \geq 2$  if  $\Pi(n_0) = n_0 + \beta_m n_0^m + O(n_0^{m+1})$  with  $\beta_m \neq 0$ . Then, one has the following result for limit cycles.

**Theorem 46** [50] *Let  $\gamma$  be a periodic orbit of the analytic system (3) and let  $V$  be an analytic inverse integrating factor defined in a neighborhood of  $\gamma$ .*

- (a) *If  $\gamma$  is a limit cycle of multiplicity  $m$ , then  $V$  has vanishing multiplicity  $m$  on  $\gamma$ .*
- (b) *If  $V$  has vanishing multiplicity  $m$  on  $\gamma$ , then  $\gamma$  is a limit cycle of multiplicity  $m$  or it belongs to a continuum of periodic orbits.*

Since the Poincaré map of a periodic orbit is an analytic function and the multiplicity of a limit cycle is a natural number, the following corollary is obtained.

**Corollary 47** [50] *Let  $\gamma$  be a periodic orbit of the analytic system (3) and let  $V$  be an inverse integrating factor of class  $\mathcal{C}^1$  defined in a neighborhood of  $\gamma$ . We take the*

change to curvilinear coordinates  $x(s, n) = \varphi(s) - n\psi'(s)$ ,  $y(s, n) = \psi(s) + n\varphi'(s)$  defined in a neighborhood of  $\gamma$ . If we have that the leading term in the following development around  $n = 0$ :

$$V(x(s, n), y(s, n)) = n^\rho v(s) + o(n^\rho),$$

where  $v(s) \not\equiv 0$  is such that either  $\rho = 0$  or  $\rho > 1$  and  $\rho$  is not a natural number, then  $\gamma$  belongs to a continuum of periodic orbits.

A regular orbit  $\phi(t) = (x(t), y(t))$  of (3) is called a *homoclinic orbit* if  $\phi(t) \rightarrow p_0$  as  $t \rightarrow \pm\infty$  for some singular point  $p_0$ . A *homoclinic loop* is the union  $\Gamma = \phi(t) \cup \{p_0\}$ . We assume that  $p_0$  is a hyperbolic saddle, that is, a critical point of system (3) such that the eigenvalues of the Jacobian matrix  $D\mathcal{X}(p_0)$  are both real, different from zero and of contrary sign. We remark that this type of graphics always has associated (maybe only its inner or outer neighborhood) a Poincaré return map  $\Pi : \Sigma \rightarrow \Sigma$  with  $\Sigma$  any local transversal section through a regular point of  $\Gamma$ . We will assume that  $\Gamma$  is a compact invariant set. A goal in [50] is to study the cyclicity of the described homoclinic loop  $\Gamma$  in terms of the vanishing multiplicity of an inverse integrating factor. Roughly speaking, the *cyclicity* of  $\Gamma$  is the maximum number of limit cycles which bifurcate from it under an analytic perturbation of the analytic system (3). Before state the result for homoclinic loops, we recall briefly that the first saddle quantity is  $\alpha_1 = \text{div } \mathcal{X}(p_0)$  and it classifies the point  $p_0$  between being strong (when  $\alpha_1 \neq 0$ ) or weak (when  $\alpha_1 = 0$ ). If  $p_0$  is a weak saddle point, the saddle quantities are the obstructions for it to be analytically orbitally linearizable. In order to define the next saddle quantities associated to  $p_0$ , we translate the saddle-point  $p_0$  to the origin of coordinates and we make a linear change of variables so that its unstable (resp. stable) separatrix has the horizontal (resp. vertical) direction at the origin. Let  $p_0$  be a weak hyperbolic saddle point situated at the origin of coordinates and whose associated eigenvalues are taken to be  $\pm 1$  by a rescaling of time, if necessary. Then, it is well known the existence of an analytic near-identity change of coordinates that brings the system into:

$$\begin{aligned} \dot{x} &= x + \sum_{i=1}^{k-1} a_i x^{i+1} y^i + a_k x^{k+1} y^k + \dots, \\ \dot{y} &= -y - \sum_{i=1}^{k-1} a_i x^i y^{i+1} - b_k x^k y^{k+1} + \dots, \end{aligned} \tag{23}$$

with  $a_k - b_k \neq 0$  and where the dots denote terms of higher order. The first non-vanishing saddle quantity is defined by  $\alpha_{k+1} := a_k - b_k$ .

**Theorem 48** [50] *Let  $\Gamma$  be a compact homoclinic loop through the hyperbolic saddle  $p_0$  of the analytic system (3) whose Poincaré return map is not the identity. Let*



$V$  be an analytic inverse integrating factor defined in a neighborhood of  $\Gamma$  with vanishing multiplicity  $m$  over  $\Gamma$ . Then,  $m \geq 1$  and the first possible non-vanishing saddle quantity is  $\alpha_m$ . Moreover,

- (i) the cyclicity of  $\Gamma$  is  $2m - 1$ , if  $\alpha_m \neq 0$ ,
- (ii) the cyclicity of  $\Gamma$  is  $2m$ , otherwise.

In addition, in [50] it is described one obstruction to the existence of an analytic inverse integrating factor defined in a neighborhood of certain homoclinic loops. First of all, we recall some concepts. By an affine change of coordinates, in a neighborhood of a hyperbolic saddle, any analytic system can be written as  $\dot{x} = \lambda x + f(x, y)$ ,  $\dot{y} = \mu y + g(x, y)$ , where  $f$  and  $g$  are analytic in a neighborhood of the origin with lowest terms at least of second order and  $\mu < 0 < \lambda$ . This hyperbolic saddle is analytically *orbitally linearizable* if there exists an analytic near-identity change of coordinates transforming the system to  $\dot{x} = \lambda x h(x, y)$ ,  $\dot{y} = \mu y h(x, y)$  with  $h(0, 0) = 1$ . On the other hand, when  $\mu/\lambda = -q/p \in \mathbb{Q}^-$  with  $p$  and  $q$  natural and coprime numbers, the saddle is called  $p : q$  resonant.

**Proposition 49** [50] *Suppose that the analytic system (3) has a homoclinic loop  $\Gamma$  through the hyperbolic saddle point  $p_0$  which is not orbitally linearizable,  $p : q$  resonant and strong ( $p \neq q$ ). Then, there is no analytic inverse integrating factor  $V(x, y)$  defined in a neighborhood of  $\Gamma$ .*

### 8.3 Generalized Hopf Bifurcation

Let us consider a planar real system (3),  $\dot{x} = P(x, y)$ ,  $\dot{y} = Q(x, y)$  and suppose that it is analytic near an isolated monodromic singular point  $p_0$  which we assume to be at the origin. We associate to system (3) the vector field  $\mathcal{X}_0 = P(x, y)\partial_x + Q(x, y)\partial_y$ . We consider an analytic perturbation of system (3) of the form:

$$\dot{x} = P(x, y) + \bar{P}(x, y, \varepsilon), \quad \dot{y} = Q(x, y) + \bar{Q}(x, y, \varepsilon), \quad (24)$$

where  $\varepsilon \in \mathbb{R}^p$  is the perturbation parameter,  $0 < \|\varepsilon\| \ll 1$  and the functions  $\bar{P}(x, y, \varepsilon)$  and  $\bar{Q}(x, y, \varepsilon)$  are analytic for  $(x, y) \in \mathcal{U}$ , analytic in a neighborhood of  $\varepsilon = 0$  and  $\bar{P}(x, y, 0) = \bar{Q}(x, y, 0) \equiv 0$ . We associate to this perturbed system (24) the vector field  $\mathcal{X}_\varepsilon = (P(x, y) + \bar{P}(x, y, \varepsilon))\partial_x + (Q(x, y) + \bar{Q}(x, y, \varepsilon))\partial_y$ .

We say that a limit cycle  $\gamma_\varepsilon$  of system (24) *bifurcates from the origin* if it tends to the origin (in the Hausdorff distance) as  $\varepsilon \rightarrow 0$ . We are interested in giving a sharp upper bound for the number of limit cycles which can bifurcate from the origin  $p_0$  of system (3) under any analytic perturbation with a finite number  $p$  of parameters. The word sharp means that there exists a system of the form (24) with exactly that number of limit cycles bifurcating from the origin, that is, the

upper bound is realizable. This sharp upper bound is called the *cyclicity* of the origin  $p_0$  of system (3) and will be denoted by  $\text{Cycl}(\mathcal{X}_\varepsilon, p_0)$  all along this section.

In [51] we consider systems of the form (3) where the origin  $p_0$  is a focus singular point of the following three types: non-degenerate, degenerate without characteristic directions and nilpotent. The results of [51] do not establish that the cyclicity of this type of singular points is finite but give an effective procedure to study it. In the three mentioned types of focus points, we will consider a change to (generalized) polar coordinates which embed the neighborhood  $\mathcal{U}$  of the origin into a cylinder  $C = \{(r, \theta) \in \mathbb{R} \times \mathcal{S}^1 : |r| < \delta\}$  for a certain sufficiently small value of  $\delta > 0$ . This change to polar coordinates is a diffeomorphism in  $\mathcal{U} - \{(0, 0)\}$  and transforms the origin of coordinates to the circle of equation  $r = 0$ . In these new coordinates, system (3) can be seen as a differential equation over the cylinder  $C$  of the form:

$$\frac{dr}{d\theta} = \mathcal{F}(r, \theta), \quad (25)$$

where  $\mathcal{F}(r, \theta)$  is an analytic function in  $C$ . We consider an inverse integrating factor  $V(r, \theta)$  of equation (25), that is, a function  $V : C \rightarrow \mathbb{R}$  of class  $\mathcal{C}^1(C)$ , which is non locally null and which satisfies the following partial differential equation:

$$\frac{\partial V(r, \theta)}{\partial \theta} + \frac{\partial V(r, \theta)}{\partial r} \mathcal{F}(r, \theta) = \frac{\partial \mathcal{F}(r, \theta)}{\partial r} V(r, \theta).$$

We remark that since  $V(r, \theta)$  is a function defined over the cylinder  $C$  it needs to be  $T$ -periodic in  $\theta$ , where  $T$  is the minimal positive period of the variable  $\theta$ , that is, we consider the circle  $\mathcal{S}^1 = \mathbb{R}/[0, T]$ . The function  $V(r, \theta)$  is smooth ( $\mathcal{C}^\infty$ ) and non-flat in  $r$  in a neighborhood of  $r = 0$ .

Let us consider the Taylor expansion of the function  $V(r, \theta)$  around  $r = 0$ :  $V(r, \theta) = v_m(\theta) r^m + \mathcal{O}(r^{m+1})$ , where  $v_m(\theta) \not\equiv 0$  for  $\theta \in \mathcal{S}^1$  and  $m$  is an integer number with  $m \geq 0$ . We say that  $m$  is the *vanishing multiplicity* of  $V(r, \theta)$  on  $r = 0$ . The uniqueness of  $V(r, \theta)$  implies that the number  $m$  corresponding to the vanishing multiplicity of  $V(r, \theta)$  on  $r = 0$  is well-defined.

We consider a system (3) of the form:

$$\dot{x} = p_d(x, y) + P_{d+1}(x, y), \quad \dot{y} = q_d(x, y) + Q_{d+1}(x, y), \quad (26)$$

where  $d \geq 1$  is an odd number,  $p_d(x, y)$  and  $q_d(x, y)$  are homogeneous polynomials of degree  $d$  and  $P_{d+1}(x, y), Q_{d+1}(x, y) \in \mathcal{O}(\|(x, y)\|^{d+1})$ . We assume that  $p_d^2(x, y) + q_d^2(x, y) \not\equiv 0$ . A *characteristic direction* for the origin of system (26) is a linear factor in  $\mathbb{R}[x, y]$  of the homogeneous polynomial  $xq_d(x, y) - yp_d(x, y)$ . If there are no characteristic directions, then the origin is a monodromic singular point of system (26), that is, it is either a center or a focus.

In relation with system (24), an analytic perturbation field  $(\bar{P}(x, y, \varepsilon), \bar{Q}(x, y, \varepsilon))$  is said to have subdegree  $s$  if  $(\bar{P}(x, y, \varepsilon), \bar{Q}(x, y, \varepsilon)) = \mathcal{O}(\|(x, y)\|^s)$ . In this case, we denote by  $\mathcal{X}_\varepsilon^{[s]} = (P(x, y) + \bar{P}(x, y, \varepsilon))\partial_x + (Q(x, y) + \bar{Q}(x, y, \varepsilon))\partial_y$  the vector field associated to such a perturbation.

**Theorem 50** [51] *We assume that the origin  $p_0$  of system (26) is monodromic and without characteristic directions. Take polar coordinates  $x = r \cos \theta$ ,  $y = r \sin \theta$  and let  $V(r, \theta)$  be an inverse integrating factor of the corresponding equation (25) which has a Laurent expansion in a neighborhood of  $r = 0$  of the form  $V(r, \theta) = v_m(\theta) r^m + \mathcal{O}(r^{m+1})$ , with  $v_m(\theta) \neq 0$  and  $m \in \mathbb{Z}$ .*

- (i) *If  $m \leq 0$  or  $m$  is even, then the origin of system (26) is a center.*
- (ii) *If the origin of system (26) is a focus, then  $m \geq 1$ ,  $m$  is an odd number and the cyclicity  $\text{Cycl}(\mathcal{X}_\varepsilon, p_0)$  of the origin of system (26) satisfies  $\text{Cycl}(\mathcal{X}_\varepsilon, p_0) \geq (m+d)/2 - 1$ . In this case  $m$  is the vanishing multiplicity of  $V(r, \theta)$  on  $r = 0$ .*
  - (ii.1) *If, moreover, the focus is non-degenerate ( $d = 1$ ), then the aforementioned lower bound is sharp, that is,  $\text{Cycl}(\mathcal{X}_\varepsilon, p_0) = (m - 1)/2$ .*
  - (ii.2) *If only perturbations whose subdegree is greater than or equal to  $d$  are considered, then the maximum number of limit cycles which bifurcate from the origin is  $(m - 1)/2$ , that is,  $\text{Cycl}(\mathcal{X}_\varepsilon^{[d]}, p_0) = (m - 1)/2$ .*

**REMARK 51** *From the proof of Theorem 50, it follows that if there exists an inverse integrating factor  $V_0(x, y)$  of system (26) such that  $V_0(r \cos \theta, r \sin \theta)/r^d$  has a Laurent expansion in a neighborhood of  $r = 0$ , then the exponents of the leading terms of  $V_0(r \cos \theta, r \sin \theta)/r^d$  and  $V(r, \theta)$  coincide. Thus, the vanishing multiplicity  $m$  can be computed without passing the system to polar coordinates.*

We assume that the origin of system (26) is a focus without characteristic directions and that the vanishing multiplicity of an inverse integrating factor on it is  $m$ . If system (26) is written as  $\dot{x} = P(x, y)$  and  $\dot{y} = Q(x, y)$ , then the system:

$$\dot{x} = P(x, y) + x K(x, y, \varepsilon), \quad \dot{y} = Q(x, y) + y K(x, y, \varepsilon), \quad (27)$$

where  $L := (m + d)/2 - 1$  and  $K(x, y, \varepsilon) = \sum_{i=0}^{k-1} \varepsilon^{k-i} a_i (x^2 + y^2)^{i + \frac{d-1}{2}}$ , has at least  $(m + d)/2 - 1$  limit cycles bifurcating from the origin for convenient values of the real parameters  $a_i$ . We recall that both  $m$  and  $d$  are odd and  $d \geq 1$ ,  $m \geq 1$ .

We say that the origin of system (3) is a *nilpotent singular point* if it is a degenerate singularity that can be written as:

$$\dot{x} = y + P_2(x, y), \quad \dot{y} = Q_2(x, y), \quad (28)$$

with  $P_2(x, y)$  and  $Q_2(x, y)$  analytic functions near the origin without constant nor linear terms. The following theorem is due to Andreev [1] and it solves the monodromy problem for the origin of system (28).

**Theorem 52** [1] *Let  $y = F(x)$  be the solution of  $y + P_2(x, y) = 0$  passing through  $(0, 0)$ . Define the functions  $f(x) = Q_2(x, F(x)) = ax^\alpha + \dots$  with  $a \neq 0$  and  $\alpha \geq 2$  and  $\phi(x) = (\partial P_2 / \partial x + \partial Q_2 / \partial y)(x, F(x))$ . We have that either  $\phi(x) = bx^\beta + \dots$  with  $b \neq 0$  and  $\beta \geq 1$  or  $\phi(x) \equiv 0$ . Then, the origin of (28) is monodromic if, and only if,  $a < 0$ ,  $\alpha = 2n - 1$  is an odd integer and one of the following conditions holds: (i)  $\beta > n - 1$ ; (ii)  $\beta = n - 1$  and  $b^2 + 4an < 0$ ; (iii)  $\phi(x) \equiv 0$ .*

**Definition 53** *We consider a system of the form (28) with the origin as a monodromic singular point. We define its Andreev number  $n \geq 2$  as the corresponding integer value given in Theorem 52.*

We consider system (28) and we assume that the origin is a nilpotent monodromic singular point with Andreev number  $n$ . Then, the change of variables

$$(x, y) \mapsto (x, y - F(x)), \quad (29)$$

where  $F(x)$  is defined in Theorem 52, and the scaling

$$(x, y) \mapsto (\xi x, -\xi y), \quad (30)$$

with  $\xi = (-1/a)^{1/(2-2n)}$ , brings system (28) into the following analytic form for monodromic nilpotent singularities

$$\dot{x} = y(-1 + X_1(x, y)), \quad \dot{y} = f(x) + y\phi(x) + y^2 Y_0(x, y), \quad (31)$$

where  $X_1(0, 0) = 0$ ,  $f(x) = x^{2n-1} + \dots$  with  $n \geq 2$  and either  $\phi(x) \equiv 0$  or  $\phi(x) = bx^\beta + \dots$  with  $\beta \geq n - 1$ . We remark that we have relabelled the functions  $f(x)$ ,  $\phi(x)$  and the constant  $b$  with respect to the ones corresponding to system (28).

We are going to transform system (31) to an equation over a cylinder of the form (25). The transformation depends on the Andreev number  $n$  and it is given through the *generalized trigonometric functions* defined by Lyapunov as the unique solution  $x(\theta) = \text{Cs } \theta$  and  $y(\theta) = \text{Sn } \theta$  of the following Cauchy problem

$$\frac{dx}{d\theta} = -y, \quad \frac{dy}{d\theta} = x^{2n-1}, \quad x(0) = 1, \quad y(0) = 0. \quad (32)$$

We introduce in  $\mathbb{R}^2 \setminus \{(0, 0)\}$  the change to *generalized polar coordinates*,  $(x, y) \mapsto (r, \theta)$ , defined by

$$x = r \text{Cs } \theta, \quad y = r^n \text{Sn } \theta. \quad (33)$$

We consider the following definition, which will be used in the following Theorem 55.

**Definition 54** *An analytic perturbation vector field  $(\bar{P}(x, y, \varepsilon), \bar{Q}(x, y, \varepsilon))$  is said to be  $(1, n)$ -quasihomogeneous of weighted subdegrees  $(w_x, w_y)$  if  $\bar{P}(\lambda x, \lambda^n y, \varepsilon) = \mathcal{O}(\lambda^{w_x})$  and  $\bar{Q}(\lambda x, \lambda^n y, \varepsilon) = \mathcal{O}(\lambda^{w_y})$ . In this case, we denote by  $\mathcal{X}_\varepsilon^{[w_x, w_y]} = (P(x, y) + \bar{P}(x, y, \varepsilon))\partial_x + (Q(x, y) + \bar{Q}(x, y, \varepsilon))\partial_y$  the vector field associated to such a perturbation.*

The following theorem is one of the main results of [51]. The symbol  $\lfloor x \rfloor$  denotes the integer part of  $x$ .

**Theorem 55** [51] *We assume that the origin of system (28) is monodromic with Andreev number  $n$ . Take generalized polar coordinates (33) and let  $V(r, \theta)$  be an inverse integrating factor of the corresponding equation (25) which has a Laurent expansion in a neighborhood of  $r = 0$  of the form  $V(r, \theta) = v_m(\theta)r^m + \mathcal{O}(r^{m+1})$ , with  $v_m(\theta) \not\equiv 0$  and  $m \in \mathbb{Z}$ .*

- (i) *If  $m \leq 0$  or  $m + n$  is odd, then the origin of system (28) is a center.*
- (ii) *If the origin of system (28) is a focus, then  $m \geq 1$ ,  $m + n$  is even and its cyclicity  $\text{Cycl}(\mathcal{X}_\varepsilon, p_0)$  satisfies  $\text{Cycl}(\mathcal{X}_\varepsilon, p_0) \geq (m + n)/2 - 1$ . In this case,  $m$  is the vanishing multiplicity of  $V(r, \theta)$  on  $r = 0$ .*
- (iii) *If the origin of system (31) is a focus and if only analytic perturbations of  $(1, n)$ -quasihomogeneous weighted subdegrees  $(w_x, w_y)$  with  $w_x \geq n$  and  $w_y \geq 2n - 1$  are taken into account, then the maximum number of limit cycles which bifurcate from the origin is  $\lfloor (m - 1)/2 \rfloor$ , that is,  $\text{Cycl}(\mathcal{X}_\varepsilon^{[n, 2n-1]}, p_0) = \lfloor (m - 1)/2 \rfloor$ .*

**REMARK 56** *The proof of this theorem shows that if there exists an inverse integrating factor  $V_0^*(x, y)$  of system (31) such that  $V_0^*(r \text{Cs} \theta, r^n \text{Sn} \theta)/r^{2n-1}$  has a Laurent expansion in a neighborhood of  $r = 0$ , then the exponents of the leading terms of  $V_0^*(r \text{Cs} \theta, r^n \text{Sn} \theta)/r^{2n-1}$  and  $V(r, \theta)$  coincide. Therefore, the value of  $m$  can be determined without performing the transformation of the system to generalized polar coordinates.*

We assume that the origin of system (31) is a focus with Andreev number  $n$  and that the vanishing multiplicity of an inverse integrating factor on it is  $m$ . If system (31) is written as  $\dot{x} = P(x, y)$  and  $\dot{y} = Q(x, y)$ , then the system:

$$\dot{x} = P(x, y) + x K(x, y, \varepsilon), \quad \dot{y} = Q(x, y) + ny K(x, y, \varepsilon), \quad (34)$$

where

$$K(x, y, \varepsilon) = \sum_{i=0}^{L-1} \varepsilon^{L-i} a_i x^{2i}$$

and  $L = (m + n)/2 - 1$ , has at least  $(m + n)/2 - 1$  limit cycles bifurcating from the origin for convenient values of the real parameters  $a_i$ . We recall that  $m$  and  $n$  have the same parity.

The following corollary establishes a necessary condition for system (28) to have an analytic inverse integrating factor  $V_0(x, y)$  defined in a neighborhood of the origin.

**Corollary 57** *We assume that the origin of system (28) is a nilpotent focus with Andreev number  $n$ , and that there exists an inverse integrating factor  $V_0(x, y)$  of (28) which is analytic in a neighborhood of the origin. Then,  $n$  is odd.*

## 8.4 Singular perturbations

As some recent research papers show, see [45], limit periodic sets containing an infinite number of critical points may have a cyclicity higher than expected. Due to the narrow relationship between limit cycles and the inverse integrating factor, the context of singular perturbations is a brand new and very interesting place to apply properties of the inverse integrating factor in order to detect limit cycles. As we have seen, for other limit periodic sets, the inverse integrating factor does not only give an alternative way to study the cyclicity but contains more information: location of limit cycles, direct computation of the cyclicity of the object under study, an explicit partial differential equation (4) which gathers all the information, ...

As far as we know, the only work where the inverse integrating factor is related with a singular perturbation problem is [79], where one-parameter families of vector fields  $\mathcal{X}_\varepsilon$  in  $\mathbb{R}^2$  of the form  $\mathcal{X}_\varepsilon = f(x, y, \varepsilon) \partial_x + \varepsilon g(x, y, \varepsilon) \partial_y$ , where  $\varepsilon \geq 0$  and  $f, g$  are analytic functions, are taken into account. The aim of the singular perturbation problems is to study the phase portrait, for  $\varepsilon$  sufficiently small, near the set of singular points of  $\mathcal{X}_0$ , that is,  $\Sigma = \{(x, y) \in \mathbb{R}^2 : f(x, y, 0) = 0\}$ . In particular, the question is to decide if  $\mathcal{X}_\varepsilon$  has a limit cycle which tends to a singular orbit of  $\mathcal{X}_0$  when  $\varepsilon \searrow 0$ . A singular orbit (also denoted as *slow-fast cycle*, see [45, 46]) is a limit periodic set of the system  $\mathcal{X}_0$ . For the vector field  $\mathcal{X}_0$ , we say that a point  $n \in \Sigma$  is *normally hyperbolic* if  $(\partial f / \partial x)(n, 0) \neq 0$ . The system of differential equations associated to  $\mathcal{X}_\varepsilon$  is

$$\dot{x} = f(x, y, \varepsilon), \quad \dot{y} = \varepsilon g(x, y, \varepsilon),$$

where the dot denotes derivation with respect to the time  $t$ . We call this system the *fast system*. By the time rescaling  $\tau = \varepsilon t$ , we get the *slow system*:

$$\varepsilon x' = f(x, y, \varepsilon), \quad y' = g(x, y, \varepsilon),$$

where  $'$  denotes derivation with respect to  $\tau$ . The *reduced problem* is defined by the slow system taking  $\varepsilon = 0$ , which gives one differential equation constrained to the slow manifold or critical curve  $\Sigma$ , that is, the reduced problem is

$$f(x, y, 0) = 0, \quad y' = g(x, y, \varepsilon).$$

The only singular orbits taken into account in [79] are the ones consisting of three pieces of smooth curves; an orbit of the reduced problem starting at a normally hyperbolic point  $n_1 \in \Sigma$ , an orbit of the reduced problem ending at a normally hyperbolic point  $n_2 \in \Sigma$  and an orbit of the fast problem connecting the two previous ones. The main results are the following.

**Theorem 58** [79] *Consider  $\varepsilon_0 > 0$  and  $V_\varepsilon(x, y)$  an inverse integrating factor of  $X_\varepsilon$ , that is  $\mathcal{X}_\varepsilon(V_\varepsilon(x, y)) = \operatorname{div} \mathcal{X}_\varepsilon V_\varepsilon(x, y)$ , defined in an open set  $\mathcal{U} \subseteq \mathbb{R}^2$  for any  $0 \leq \varepsilon \leq \varepsilon_0$ . Let  $\Gamma \subset \mathcal{U}$  be a singular orbit and  $\Gamma_\varepsilon$  be a limit cycle of  $\mathcal{X}_\varepsilon$  in  $\mathcal{U}$  for  $\varepsilon \in (0, \varepsilon_0)$ , with  $\Gamma_\varepsilon \rightarrow \Gamma$ , according to the Hausdorff distance. Then  $V_0(\Gamma) = 0$ .*

**Corollary 59** [79] *Consider  $V_\varepsilon(x, y)$  an inverse integrating factor of  $X_\varepsilon$  as in Theorem 58. If the level zero of the function  $V_0(x, y)$  does not contain a closed curve, then there exists  $\varepsilon_0 > 0$  such that  $\mathcal{X}_\varepsilon$  does not present a limit cycle for  $0 < \varepsilon < \varepsilon_0$  in  $\mathcal{U}$ .*

As an application of these results, the following examples are given in [79]. The following vector fields present no limit cycles because the corresponding inverse integrating factors have no closed curves in their level zero sets.

- The vector field  $\mathcal{X}_\varepsilon = (y^2 - x^2)\partial_x + \varepsilon x^2 \partial_y$  has the inverse integrating factor  $V_\varepsilon(x, y) = y^3 - yx^2 - x^3\varepsilon$ .
- The vector field  $\mathcal{X}_\varepsilon = (y - x^2)\partial_x + \varepsilon x \partial_y$  has the inverse integrating factor  $V_\varepsilon(x, y) = -y + x^2 + (1/2)\varepsilon$ .
- The vector field  $\mathcal{X}_\varepsilon = (-y + x^2)\partial_x + \varepsilon x \partial_y$  has the inverse integrating factor  $V_\varepsilon(x, y) = y - x^2 + (1/2)\varepsilon$ .

## 9 Some generalizations

### 9.1 The inverse Jacobi multiplier

Inverse Jacobi multipliers are a natural generalization of inverse integrating factors to  $n$ -dimensional dynamical systems with  $n \geq 3$ . In [4], it is developed the theory of inverse Jacobi multiplier from its beginning in the formal methods of integration of ordinary differential equations to modern applications.

In this section we will assume that  $\mathcal{X} = \sum_{i=1}^n X_i(x_1, \dots, x_n) \partial_{x_i}$  is a  $\mathcal{C}^1$  vector field defined in the open set  $\mathcal{U} \subseteq \mathbb{R}^n$ . A  $\mathcal{C}^1$  function  $V : \mathcal{U} \rightarrow \mathbb{R}$  is said to be an inverse Jacobi multiplier for the vector field  $\mathcal{X}$  in  $\mathcal{U}$  when  $V$  solves in  $\mathcal{U}$  the linear first order partial differential equation  $\mathcal{X}V = V \operatorname{div} \mathcal{X}$ . The first appearance of these multipliers occurs in the works of C.G.J. Jacobi, about the middle of the past century. Many properties of inverse integrating factors for the planar case ( $n = 2$ ) are inherited by inverse Jacobi multiplier. We list some of them:

- If the change of coordinates  $y = \phi(x)$  is introduced, then  $W(y) = (V \circ \phi^{-1})(y) \det\{D\phi(\phi^{-1}(y))\}$  is an inverse multiplier of the transformed vector field  $\phi_* \mathcal{X}$ .
- Let  $V_1$  and  $V_2$  be two linearly independent inverse Jacobi multipliers of  $\mathcal{X}$  defined in  $\mathcal{U}$ . If  $V_1(x) \neq 0$  for all  $x \in \mathcal{U}$ , then the ratio  $V_2/V_1$  is a first integral of  $\mathcal{X}$  in  $\mathcal{U}$ .
- One can use local Lie groups of transformations to find inverse Jacobi multipliers as follows. Assume  $\mathcal{X}$  admits in  $\mathcal{U}$  a  $(n - 1)$ -parameter local Lie group of transformations with infinitesimal generators  $\{\mathcal{Y}_1, \dots, \mathcal{Y}_{n-1}\}$ . Then, an inverse Jacobi multiplier  $V$  for  $\mathcal{X}$  in  $\mathcal{U}$  is furnished by the determinant  $V = \det\{\mathcal{X}, \mathcal{Y}_1, \dots, \mathcal{Y}_{n-1}\}$ .
- Let  $\{\mathcal{Y}_1, \dots, \mathcal{Y}_{n-1}\}$  be the generators of  $n - 1$  local Lie groups of symmetries admitted by  $\mathcal{X}$  in  $\mathcal{U}$ . Then, the inverse multiplier  $V = \det\{\mathcal{X}, \mathcal{Y}_1, \dots, \mathcal{Y}_{n-1}\}$  vanishes on every invariant solution of  $\mathcal{X}$  contained in  $\mathcal{U}$ . Recall here that an invariant solution of  $\mathcal{X}$  corresponding to the group  $G$  is defined to be an integral curve of  $\mathcal{X}$  which is invariant under the action of  $G$ .
- Let  $p_0 \in \mathcal{U}$  be an isolated zero of an inverse Jacobi multiplier  $V$  such that  $V \geq 0$  in a neighborhood  $\mathcal{N}$  of  $p_0$ . Then,  $p_0$  is a stable (resp. unstable) singular point of  $\mathcal{X}$  provided that  $\operatorname{div} \mathcal{X} \leq 0$  (resp.  $\geq 0$ ) in  $\mathcal{N}$ . Furthermore, the stability (resp. unstableness) of  $p_0$  is asymptotic stability (resp. unstableness) provided that  $\operatorname{div} \mathcal{X} < 0$  (resp.  $> 0$ ) in  $\mathcal{N}$ .

In the following, we summarize some of the results obtained in [4]. By a *limit cycle*  $\gamma$  of  $\mathcal{X}$  we mean a  $T$ -periodic orbit which is  $\alpha$  or  $\omega$ -limit set of another orbit of  $\mathcal{X}$ . Let  $V$  be an inverse Jacobi multiplier defined in a region containing  $\gamma$ . If  $\gamma = \{\gamma(t) \in \mathcal{U} : 0 \leq t \leq T\}$ , we define

$$\Delta(\gamma) = \int_0^T \operatorname{div} \mathcal{X} \circ \gamma(t) \, dt .$$

As it is well known,  $\Delta(\gamma)$  is the sum of the characteristic exponents of the limit cycle  $\gamma$ . We recall that if  $\Delta(\gamma) > 0$  then  $\gamma$  is not orbitally stable. We will say that



$\gamma$  is a *strong* limit cycle when  $\Delta(\gamma) \neq 0$ . If, on the contrary,  $\Delta(\gamma) = 0$ , then we say that  $\gamma$  is a *weak* limit cycle.

**Theorem 60** [4] *Let  $V$  be an inverse Jacobi multiplier of  $\mathcal{X}$  defined in a region containing a limit cycle  $\gamma$  of  $\mathcal{X}$ . Then,  $\gamma$  is contained in  $V^{-1}(0)$  in the following cases: (i) if  $\gamma$  is a strong limit cycle, or (ii) if  $\gamma$  is asymptotically orbitally stable (unstable).*

**Theorem 61** [4] *Let  $V$  be a Jacobi inverse multiplier defined in a neighborhood of a limit cycle  $\gamma$  of the vector field  $\mathcal{X}$ . If  $\gamma$  is a strong limit cycle, then*

- (i)  *$V$  vanishes on  $W^s(\gamma)$ , the stable manifold of  $\gamma$ , provided that  $\Delta(\gamma) > 0$ ;*
- (ii)  *$V$  vanishes on  $W^u(\gamma)$ , the unstable manifold of  $\gamma$ , provided that  $\Delta(\gamma) < 0$ .*

The following example appears in [4]. Consider the cubic polynomial vector field in  $\mathbb{R}^3$

$$\dot{x} = \lambda(-y + xf(x, y)), \quad \dot{y} = \lambda(x + yf(x, y)), \quad \dot{z} = z, \quad (35)$$

where  $f(x, y) = 1 - x^2 - y^2$  and  $\lambda > 0$  is a real parameter. The circle  $\gamma = \{f(x, y) = 0\} \cap \{z = 0\}$  is a limit cycle of system (35) of period  $T = 2\pi/\lambda$ . In fact,  $\gamma(t) = (\cos \lambda t, \sin \lambda t, 0)$ . It is easy to compute that

$$\Delta(\gamma) = \int_0^T \operatorname{div} \mathcal{X} \circ \gamma(t) dt = \frac{2(1 - 2\lambda)}{\lambda} \pi.$$

An inverse Jacobi multiplier for this system is

$$V_1(x, y, z) = f(x, y)(x^2 + y^2)z.$$

In addition, when  $\lambda = -1/2$ , then  $V_2(x, y, z) = (x^2 + y^2)^2$  is another inverse Jacobi last multiplier for system (35).

As usual, a hyperbolic singular point  $p_0$  of a  $\mathcal{C}^1$  vector field  $\mathcal{X}$  is named a *saddle point* when the matrix  $D\mathcal{X}(p_0)$  has eigenvalues with both positive and negative real parts. Assuming that  $k$  of these real parts are positive and the remaining  $n - k$  are negative, the stable manifold theorem ensures the existence of two invariant  $\mathcal{C}^1$  manifolds  $W^u(p_0)$  and  $W^s(p_0)$  with dimensions  $\dim W^u(p_0) = k$  and  $\dim W^s(p_0) = n - k$ , such that they intersect transversally one each other in  $p_0$ .

**Theorem 62** [4] *Let  $p_0$  be a nondegenerate strong singular point of the  $\mathcal{C}^1$  vector field  $\mathcal{X}$  having an inverse Jacobi multiplier  $V$  defined in a neighborhood of  $p_0$ . Then  $V$  vanishes on  $W^u(p_0)$  (resp.  $W^s(p_0)$ ) provided that  $\operatorname{div} \mathcal{X}(p_0) < 0$  (resp.  $\operatorname{div} \mathcal{X}(p_0) > 0$ ).*

## 9.2 Time-dependent inverse integrating factors

In [54], the authors consider autonomous second order differential equations

$$\ddot{x} = w(x, \dot{x}) , \quad (36)$$

with  $w \in \mathcal{C}^\infty(\mathcal{U})$  and  $\mathcal{U} \subseteq \mathbb{R}^2$  an open set. They associated to (36) the first order planar system defined on  $\mathcal{U}$  in the usual way

$$\dot{x} = y , \quad \dot{y} = w(x, y) . \quad (37)$$

Moreover, it is associated to equations (36) and (37) the vector fields  $\mathcal{X} = \partial_t + \dot{x}\partial_x + w(x, \dot{x})\partial_{\dot{x}}$  and  $\bar{\mathcal{X}} = y\partial_x + w(x, y)\partial_y$ , respectively. A  $\mathcal{C}^1$  nonconstant function  $I(t, x, y)$  is called an *invariant* (or non-autonomous first integral) of system (37) in  $\mathcal{U}$  if it is constant along the solutions of (37). In other words,  $\mathcal{X}I \equiv 0$  must be satisfied in  $\mathcal{U}$ . Of course, we can find at most two functionally independent invariants of (37). Notice that an invariant provides information about the asymptotic behavior of the orbits.

A symmetry of (36) is a diffeomorphism  $\Phi : (t, x) \mapsto (\bar{t}, \bar{x})$  that maps the set of solutions of (36) into itself. Therefore, the symmetry condition for (36) is just  $\bar{x}'' = w(\bar{x}, \bar{x}')$ , where the prime denotes the derivative  $' = d/d\bar{t}$ . When the symmetry is a 1-parameter Lie group of point transformations  $\Phi_\epsilon$ , then  $\bar{t} = t + \epsilon\xi(t, x) + O(\epsilon^2)$ ,  $\bar{x} = x + \epsilon\eta(t, x) + O(\epsilon^2)$ , for  $\epsilon$  close to zero, and the vector field  $\mathcal{Y} = \xi(t, x)\partial_t + \eta(t, x)\partial_x$  is called the *infinitesimal generator* of the 1-parameter Lie group of point transformations  $\Phi_\epsilon$ . It is well known that the *determining equations* for Lie point symmetries can be obtained from the linearized condition

$$\mathcal{Y}^{[2]}(\ddot{x} - w(x, \dot{x})) = 0 \text{ when } \ddot{x} = w(x, \dot{x}) , \quad (38)$$

where  $\mathcal{Y}^{[2]} = \mathcal{Y} + \eta^{[1]}(t, x, \dot{x})\partial_{\dot{x}} + \eta^{[2]}(t, x, \dot{x}, \ddot{x})\partial_{\ddot{x}}$  is the so-called *second prolongation* of the infinitesimal generator  $\mathcal{Y}$  and  $\eta^{[1]}(t, x, \dot{x}) = D_t\eta - \dot{x}D_t\xi$ ,  $\eta^{[2]}(t, x, \dot{x}, \ddot{x}) = D_t\eta^{[1]} - \ddot{x}D_t\xi$  where  $D_t = \partial_t + \dot{x}\partial_x + \ddot{x}\partial_{\dot{x}}$  is the operator total derivative with respect to  $t$ . Of course, since (36) is autonomous, it always admits the generator  $\mathcal{Y} = \partial_t$  of a Lie point symmetry. Let  $\mathcal{L}_r$  denote the set of all infinitesimal generators of 1-parameter Lie groups of point symmetries of the differential equation (36). It is known that  $\mathcal{L}_r$  is a finite dimensional real Lie algebra, where we denote  $r = \dim \mathcal{L}_r$ . Moreover, for autonomous second order differential equation we have  $r \in \{1, 2, 3, 8\}$ .

For any  $\mathcal{Y}_i = \xi_i(t, x)\partial_t + \eta_i(t, x)\partial_x \in \mathcal{L}_r$ , easily one can check that the Lie bracket  $[\mathcal{X}, \mathcal{Y}_i^{[1]}] = \mu_i(t, x, \dot{x})\mathcal{X}$  where  $\mu_i(t, x, \dot{x}) = \mathcal{X}\xi_i$  and  $\mathcal{Y}_i^{[1]} = \mathcal{Y}_i + \eta_i^{[1]}(t, x, \dot{x})\partial_{\dot{x}}$  is the first prolongation of  $\mathcal{Y}_i$ . If  $r \geq 2$ , we define the functions

$$V_{ij}(t, x, \dot{x}) = \det\{\mathcal{X}, \mathcal{Y}_i^{[1]}, \mathcal{Y}_j^{[1]}\} = \begin{vmatrix} 1 & \dot{x} & w(x, \dot{x}) \\ \xi_i(t, x) & \eta_i(t, x) & \eta_i^{[1]}(t, x, \dot{x}) \\ \xi_j(t, x) & \eta_j(t, x) & \eta_j^{[1]}(t, x, \dot{x}) \end{vmatrix} \quad (39)$$

for  $i, j \in \{1, \dots, r\}$  with  $1 \leq i < j \leq r$ . The aim of the work [54] is to generalize the concept of inverse integrating factor  $V(x, y)$  of system (37) via the functions  $V_{ij}(t, x, y)$  defined in (39). In fact, in the autonomous particular case  $\partial V_{ij}/\partial t \equiv 0$ , we get that  $V_{ij}$  is just an inverse integrating factor of (37). On the contrary, when  $\partial V_{ij}/\partial t \not\equiv 0$ , in [54] it is proved that the zero-sets  $V^{-1}(0)$  and  $V_{ij}^{-1}(0)$  have similar properties. The next result provides the connection between inverse integrating factors of system (37) and the functions  $V_{ij}(t, x, y)$ .

**Proposition 63** *Assume that system (37) possesses an  $r$ -dimensional Lie point symmetry algebra with  $r \geq 2$  and define the functions  $V_{ij}(t, x, \dot{x})$  as in (39).*

- (i)  $V_{ij}$  satisfies the linear partial differential equation  $\mathcal{X}V_{ij} = V_{ij} \operatorname{div} \mathcal{X}$ , where  $\mathcal{X} = \partial_t + \dot{x}\partial_x + w(x, \dot{x})\partial_{\dot{x}}$ .
- (ii) If  $r \geq 3$  then, the ratio of any two nonzero  $V_{ij}$  is either a constant or an invariant of (37).
- (iii) If  $V_{ij} \equiv 0$ , then  $(\eta_i - y\xi_i)/(\eta_j - y\xi_j)$  is an invariant of system (37).

The next theorem is about the invariant curves of  $\bar{\mathcal{X}}$  contained in  $V_{ij}^{-1}(0)$  and give us an extension of Theorem 9 in [63] for a case with  $\partial V_{ij}/\partial t \not\equiv 0$ . We put special emphasis on periodic orbits of (37) of any kind (isolated and, therefore, limit cycles or non-isolated and so belonging to a period annulus). Recall here that a limit cycle  $\gamma := \{(x(t), y(t)) \in \mathcal{U} : 0 \leq t < T\}$  is *hyperbolic* if  $\oint_{\gamma} \operatorname{div} \bar{\mathcal{X}}(x(t), y(t)) dt \neq 0$ . On the other hand, a  $\mathcal{C}^1$  curve  $f(x, y) = 0$  defined on  $\mathcal{U}$  is invariant for  $\bar{\mathcal{X}}$  if  $\bar{\mathcal{X}}f = Kf$  for some function  $K(x, y)$  called *cofactor*.

**Theorem 64** [54] *Let  $\mathcal{U} \subset \mathbb{R}^2$  be an open set and assume that  $\ddot{x} = w(x, \dot{x})$  with  $w$  smooth in  $\mathcal{U}$  admits an  $r$ -dimensional Lie point symmetry algebra  $\mathcal{L}_r$  with  $r \geq 2$ . Consider the functions  $V_{ij}(t, x, \dot{x})$  defined in (39) for  $i, j \in \{1, \dots, r\}$  with  $1 \leq i < j \leq r$ . Suppose that  $\gamma = (x(t), y(t)) \subset \mathcal{U}$  is a  $T$ -periodic orbit of (37). Then the next statements hold:*

- (i) If  $V_{ij}(t, x, \dot{x}) = V(x, \dot{x}) \not\equiv 0$ , with  $V \in \mathcal{C}^1(\mathcal{U})$ , then  $V(x, y)$  is an inverse integrating factor of system (37) in  $\mathcal{U}$ . In particular, if  $\gamma$  is a limit cycle, then  $\gamma \subset \{V(x, y) = 0\}$ .
- (ii) If  $V_{ij}(t, x, \dot{x}) = F(t)G(x, \dot{x}) \not\equiv 0$  with non-constants  $F$  and  $G \in \mathcal{C}^1(\mathcal{U})$ , then  $\dot{F} = \alpha F$  with  $\alpha \in \mathbb{R} \setminus \{0\}$  and  $G(x, y) = 0$  is an invariant curve of system (37). Moreover, we have:
  - (ii.1) If  $\gamma \subset \{G = 0\}$  and  $G$  is analytic on  $\mathcal{U}$ , then  $G$  is not square-free, i.e.,  $G(x, y) = g^n(x, y)u(x, y)$  with a positive integer  $n > 1$  and  $g$  and  $u$  are analytic functions on  $\mathcal{U}$  satisfying  $\gamma \subset \{g = 0\}$  and  $\gamma \not\subset \{u = 0\}$ .

(ii.2) If  $\gamma \not\subset \{G = 0\}$  then  $\gamma$  is hyperbolic and  $\alpha T = \oint_{\gamma} \operatorname{div} \bar{\mathcal{X}}(x(t), y(t)) dt$ .

An immediate consequence is obtained.

**Corollary 65** [54] *Assume that  $\ddot{x} = w(x, \dot{x})$ , with  $w$  smooth in the open set  $\mathcal{U} \subseteq \mathbb{R}^2$ , admits an  $r$ -dimensional Lie point symmetry algebra  $\mathcal{L}_r$  with  $r \geq 2$ . Consider the functions  $V_{ij}(t, x, \dot{x})$  for  $i, j \in \{1, \dots, r\}$  with  $1 \leq i < j \leq r$ . If there is one  $V_{ij}(t, x, y) = F(t)G(x, y) \not\equiv 0$  with non-constants  $F$  and  $G \in \mathcal{C}^1(\mathcal{U})$ , then system (37) does not have period annulus in  $\mathcal{U}$ .*

In the sequel, we concentrate our attention in the 2-dimensional case  $\mathcal{L}_2$ . In [54] it is proved that, if  $\partial_t \in \mathcal{L}_2$ , then the autonomous or separate time-variable forms of  $V_{ij}(t, x, \dot{x})$  given in Theorem 64 are the only possibilities. Moreover, defining the domain of definition of the infinitesimal generators as the unbounded open strip  $\Xi = \{(t, x) \in \mathbb{R} \times \mathbb{X}\} \subset \mathbb{R}^2$ , one has the following result.

**Theorem 66** [54] *Assume that  $\ddot{x} = w(x, \dot{x})$  with  $w$  smooth in  $\mathcal{U} \subset \mathbb{R}^2$  admits a 2-dimensional Lie point symmetry algebra  $\mathcal{L}_2$  spanned by the  $\mathcal{C}^1(\Xi)$  vector fields  $\mathcal{Y}_1 = \partial_t$  and  $\mathcal{Y}_2$  such that  $[\mathcal{Y}_1, \mathcal{Y}_2] = c_1 \mathcal{Y}_1 + c_2 \mathcal{Y}_2$ .*

- (i) *If  $c_2 = 0$  and  $\mathcal{Y}_2 \in \mathcal{C}^2(\Xi)$ , then  $V_{12}(t, x, \dot{x}) = G(x, \dot{x})$  with  $G(x, y) = y^2[c_1 + y\alpha'(x) - \beta'(x)] + \beta(x)w(x, y)$  an inverse integrating factor of  $\bar{\mathcal{X}}$  in  $W = \mathcal{U} \cap \{\mathbb{X} \times \mathbb{R}\}$  provided that  $G \not\equiv 0$ . Moreover, for analytic vector fields  $\mathcal{Y}_2$  in  $\Xi$ ,  $\bar{\mathcal{X}}$  has no limit cycles in  $W$ .*
- (ii) *If  $c_2 \neq 0$  then, changing the basis of  $\mathcal{L}_2$  such that  $[\bar{\mathcal{Y}}_1, \bar{\mathcal{Y}}_2] = \bar{\mathcal{Y}}_1$ , we have that  $\bar{V}_{12}(t, x, \dot{x}) = \exp(c_2 t) \bar{G}(x, \dot{x})$  with  $\bar{G}(x, \dot{x}) = \dot{x}[c_2 \dot{x} \alpha(x) - c_2 \beta(x) + \dot{x}^2 \alpha'(x) - \dot{x} \beta'(x)] + \beta(x)w(x, \dot{x})$ . In addition,  $\partial w / \partial x \equiv 0$  or  $\beta(x) \equiv 0$ . If  $\bar{G} \not\equiv 0$  and  $\mathcal{U}$  is a simply connected domain, then  $\bar{\mathcal{X}}$  has no periodic orbits in  $\mathcal{U}$  and all the  $\alpha$  or  $\omega$ -limit sets of  $\bar{\mathcal{X}}$  are contained in the invariant curve  $\bar{G}(x, y) = 0$  of  $\bar{\mathcal{X}}$ .*

As an application of these results to polynomial Liénard systems, in [54] it is proved the next theorem.

**Theorem 67** [54] *The polynomial Liénard differential equation  $\ddot{x} + f(x)\dot{x} + g(x) = 0$  with  $f, g \in \mathbb{R}[x]$  having a  $r$ -dimensional Lie point symmetry algebra  $\mathcal{L}_r$  with  $r \geq 2$  has no limit cycles in  $\mathbb{R}^2$ .*

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